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Analysis of the Sign of the Solution for Certain Second-Order Periodic Boundary Value Problems with Piecewise Constant Arguments

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Abstract: We find sufficient conditions for the unique solution of certain second-order boundary value problems to have a constant sign. To this purpose, we use the expression in terms of a Green's function of the unique solution for impulsive linear periodic boundary value problems associated with second-order differential equations with a functional dependence, which is a piecewise constant function. Our analysis lies in the study of the sign of the Green's function.

Keywords: second-order differential equations; piecewise constant functional dependence; boundary value problems; Green's function; comparison results; positive solutions

MSC: 34K06; 34K10

1. Introduction

Delay differential equations find interesting applications in fields like biology, fisiology, physics, etc. Some interesting and recent applications of this type of equations appear, for instance, when proposing a mathematical model for electrohydraulic servomechanisms [1], absorption complexities in pharmacokinetics [2], the approximation of Fitzhugh–Nagumo and Hodgkin–Huxley models for action potential generation in excitable cells [3], tick population with diapause [4], or thermal models for variable pipe flow [5].

The study of the solutions' properties to such kind of models is a relevant area of study [6]. In particular, the existence of solutions with a constant sign is important from the point of view of the interpretation of the biological models. Indeed, the non-negativity of the solutions [7] is essential for the coherence between mathematical expressions and the magnitudes that, in this context, normally refer to positive values such as the number of individuals in a certain population or the concentration of a certain substance. Recent work [8] deepens the problem of the existence of non-negative solutions to linear autonomous functional differential equations. See also [9] for order preserving evolution operators of functional differential equations.

This paper is devoted to delay differential equations with piecewise constant delay given by the greatest integer function. The appearance of such kind of delays can be interpreted as the dependence of the rate of change of the variable of interest not only on the present state but also on some previous (equidistributed with respect to time) memorized values of the state (see [10], where a Liénard-type differential equation with piecewise constant delays is considered).

The study of second-order differential equations independent of the first-order derivative x' with a functional dependence given by a piecewise constant argument goes back to references [11–15]. Here, the functional dependence considered is the greatest integer function. In [16], the authors consider boundary value problems for second-order differential equations with deviating arguments and, in [17], the uniqueness issue is addressed for second-order linear functional differential equations.

In particular, the periodic boundary value problem:

$$\begin{cases} -x''(t) = f(t, x(t), x([t])), \, t \in J = [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

is studied in [15], where $T > 0$, and $[t]$ denotes the floor of t, that is, the greatest integer less than or equal to t. The references [18,19] are devoted to similar problems for first-order differential equations. In order to consider a problem where a delay term also affects the derivative of the function, the existence of solutions for second-order functional differential equations with piecewise constant arguments of the form:

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = \sigma(t), \, t \in J = [0, T], \\ x(0) = x(T), \\ x'(0) = x'(T) + \lambda, \end{cases} \tag{1}$$

is considered in [20] by studying the existence of solutions for the impulsive linear periodic boundary value problems:

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = 0, \, t \in \mathbb{R}, \\ x(0) = x(T), \\ x'(0^-) = x'(T^+), \\ x'(s^+) = x'(s^-) + 1, \end{cases} \tag{2}$$

where $s \in J$, a, b, c, d, $\lambda \in \mathbb{R}$, $T > 0$, and $x'(z^-)$, $x'(z^+)$ denote, respectively, the one-sided left and right limits of x' at the point $z \in J$.

Although the expression of the solution to (1) can be obtained directly from the solution of (2), which represents the Green's function, it is quite tedious to find sufficient conditions for the Green's function to have a constant sign. In this paper, we prove the existence of non-negative (resp. non-positive) solutions for (1) under the appropriate conditions.

Some other references dealing with similar problems are [21], where the existence and stability of periodic solutions for a quasi-linear differential equation with arguments which are piecewise constant functions is studied, or [22], where the stability of the null solution for the corresponding quasilinear equation is analyzed. In [23], a similar study is made to obtain the Green's function to express the unique solution to a second-order nonautonomous differential equation with piecewise constant arguments.

The objectives of the main results in the paper are summarized as follows:

- Theorems 1–3 give sufficient conditions for the unique solution to problem (2) with $s = 0$ or $s = T$ to have a constant sign. The technical Lemmas A2–A6 allow to interpret two of the sufficient conditions directly in terms of the coefficients of the linear problem a, b, c, d (see Remarks 1 and 2). The other condition is algebraically manipulated in Lemma 1, Remark 3, Lemma 2, and Remark 4;
- An analogous study can be made for $0 < s < T$ with $s \in \mathbb{Z}^+$, see Theorem 4, Lemma 3, Theorem 5, and Remark 5;
- To improve significantly these first results, it is important to determine what planar regions of the type:

$$\mathcal{S} := \{(x, y) \in \mathbb{R}^2 \, : \, x \geq 0, \, y \geq -Mx\},$$

for a certain $M > 0$, are such that their elements are mapped into the interval $[0, +\infty)$ when scalar multiplied by a certain vector $(h_1(z), h_2(z))$, which will be defined in relation with the differential problem. Lemma 4 is helpful to give an answer, therefore, some of the results are extended in the sense that the conditions required are reduced to some properties on the sign of functions h_1 and h_2, and the fact that certain iterates remain in $\mathcal{S}$ (see Theorems 6 and 7);

- In order to study the sign of the solution to (1), we need to obtain similar results for problem (2) when $0 < s < T$, $s \in (n, n+1)$, for some $n \in \mathbb{Z}^+$ (see Theorem 8). Remarks 6–8 provide some translations of the sufficient conditions given by using the technical lemmas in the appendices or direct manipulation;
- Finally, Theorems 9 and 10 provide sufficient conditions for the unique solution to problem (1) to have a constant sign on J.

Our study allows the establishment of some comparison results for boundary value problems related to second-order functional differential equations, allowing the application of some techniques to study the existence of solution to second-order nonlinear problems.

The paper is organized as follows. In Section 2, the main results on the existence of non-negative (resp. non-positive) solutions to problems (2) and (1) are given. These results are based on some preliminary and technical results that are placed on the appendices for simplicity. In Section 3, some examples are shown to illustrate the applicability of the main results. At the end, it is possible to find Appendices A–C. The first one is devoted to recall some preliminary results from [20], which are essential to the procedure since they provide the expression of the solution to the problems of interest; the second and third ones focus on the properties of the sign of three auxiliary functions, namely h_1, h_2, and g, and their derivatives, which is crucial to decide the sign of the Green's function to problem (1). The proofs of these auxiliary lemmas have been written in appendices in order to improve the readability of the work, focusing the interest in the main results.

2. Main Results

We start by introducing some additional notation that is essential to our procedure. We denote $\mathbb{Z}^+ := \mathbb{N} \cup \{0\}$. Define the functions h_1, h_2, and g by the respective expressions:

$$h_1(s) = \begin{cases} 1 - \dfrac{d}{a}s + \dfrac{d}{a^2}(1 - e^{-as}), & \text{if } b = 0,\ a \neq 0, \\[2ex] 1 - \dfrac{d}{2}s^2, & \text{if } b = 0,\ a = 0, \\[2ex] \left(1 + \dfrac{d}{b}\right)\left(1 + \dfrac{a}{2}s\right)e^{-\frac{a}{2}s} - \dfrac{d}{b}, & \text{if } b \neq 0,\ a^2 = 4b, \\[2ex] \left(1 + \dfrac{d}{b}\right)\dfrac{\beta e^{\alpha s} - \alpha e^{\beta s}}{\beta - \alpha} - \dfrac{d}{b}, & \text{if } b \neq 0,\ a^2 > 4b, \\[2ex] \left(1 + \dfrac{d}{b}\right)e^{-\frac{a}{2}s}\left\{\cos\sqrt{b - \frac{a^2}{4}}s + \frac{a}{2\sqrt{b - \frac{a^2}{4}}}\sin\sqrt{b - \frac{a^2}{4}}s\right\} - \dfrac{d}{b}, & \text{if } b \neq 0,\ a^2 < 4b, \end{cases}$$

$$h_2(s) = \begin{cases} \frac{1}{a}\left(1 - e^{-as} - cs + \frac{c}{a}(1 - e^{-as})\right), & \text{if } b = 0,\ a \neq 0, \\ s - \frac{c}{2}s^2, & \text{if } b = 0,\ a = 0, \\ e^{-\frac{a}{2}s}\left[\frac{c}{b}\left(1 + \frac{a}{2}s\right) + s\right] - \frac{c}{b}, & \text{if } b \neq 0,\ a^2 = 4b, \\ \frac{\left(\frac{\beta c}{b} - 1\right)e^{\alpha s} + \left(1 - \frac{\alpha c}{b}\right)e^{\beta s}}{\beta - \alpha} - \frac{c}{b}, & \text{if } b \neq 0,\ a^2 > 4b, \\ e^{-\frac{a}{2}s}\left\{\frac{c}{b}\cos\sqrt{b - \frac{a^2}{4}}s + \frac{1 + \frac{ac}{2b}}{\sqrt{b - \frac{a^2}{4}}}\sin\sqrt{b - \frac{a^2}{4}}s\right\} - \frac{c}{b}, & \text{if } b \neq 0,\ a^2 < 4b, \end{cases}$$

$$g(z) = \begin{cases} \frac{1 - e^{-az}}{a}, & \text{if } b = 0,\ a \neq 0, \\ z, & \text{if } b = 0,\ a = 0, \\ ze^{-\frac{a}{2}z}, & \text{if } b \neq 0,\ a^2 = 4b, \\ \frac{e^{\beta z} - e^{\alpha z}}{\beta - \alpha}, & \text{if } b \neq 0,\ a^2 > 4b, \\ e^{-\frac{a}{2}z}\frac{\sin\sqrt{b - \frac{a^2}{4}}z}{\sqrt{b - \frac{a^2}{4}}}, & \text{if } b \neq 0,\ a^2 < 4b. \end{cases} \tag{3}$$

In the previous definitions, we denote, for $b \neq 0$, $a^2 > 4b$,

$$\alpha := -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b},\ \beta := -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}.$$

For $s \in J$, we consider E_s the class of functions $y : \mathbb{R} \longrightarrow \mathbb{R}$ which satisfy the conditions:

(i) y continuous for $t \in \mathbb{R}$,
(ii) y' continuous for $t \in \mathbb{R} \setminus \{s\}$, and there exist $y'(s^-), y'(s^+) \in \mathbb{R}$,
(iii) y'' exists and is continuous for $t \in [n, n+1) \setminus \{s\}$, $n \in \mathbb{Z}$, and there exist $y''(s^-)$, $y''(s^+)$, $y''(n^-) \in \mathbb{R}$, $\forall n \in \mathbb{Z}$.

In condition iii), $y''(z^-)$, $y''(z^+)$ denote, respectively, the one-sided left and right limits of y'' at the point $z \in J$.

Definition 1 (c.f. Definition 1, [20]). *For $s \in J$, $y : \mathbb{R} \longrightarrow \mathbb{R}$ is a solution to problem (2) if $y \in E_s$ and satisfies conditions in (2), taking $y'(s) = y'(s^+)$ and $y''(t) = y''(t^+)$, for all $t \in \mathbb{Z} \cup \{s\}$.*

Recall that $[T]$ is the integer part of T, i.e., the greatest integer k, with $k \leq T$.

Definition 2 (c.f. Definition 2, [20]). *Let:*

$$\Lambda := \{y : J \longrightarrow \mathbb{R} \,:\, y \ \textit{continuous in } J \setminus \{1, 2, \ldots, [T]\}, \\ \textit{and there exist } y(n^-) \in \mathbb{R},\ y(n^+) = y(n),\ \forall n \in \{1, 2, \ldots, [T]\}\},$$

and

$$E := \{x : J \longrightarrow \mathbb{R} \,:\, x, x' \textit{ are continuous and } x'' \in \Lambda\}.$$

Definition 3 (c.f. Definition 3, [20]). *A function x is a solution to (1) if $x \in E$ and satisfies conditions in (1), taking $x''(n) = x''(n^+)$, $\forall n \in \{0, 1, 2, \ldots, [T]\}$.*

For $z \in [0,1]$, we denote by $H(z)$ the matrix:

$$H(z) := \begin{pmatrix} h_1(z) & h_2(z) \\ h_1'(z) & h_2'(z) \end{pmatrix},$$

and $C := H(1)$, so that:

$$C = H(1) = \begin{pmatrix} C_1 & C_2 \\ C_1' & C_2' \end{pmatrix}, \quad H(T-[T]) = \begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \\ h_1'(T-[T]) & h_2'(T-[T]) \end{pmatrix}.$$

Consider also I the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem 1. *Assume that condition:*

$$det\left(I - H(T-[T])C^{[T]}\right) \neq 0 \tag{4}$$

holds, that is, the matrix:

$$I - H(T-[T])\, C^{[T]} \text{ is invertible.}$$

For $s = 0$ or $s = T$, the unique solution to problem (2) is non-negative if the following conditions hold:

(i) $h_1 \geq 0$, $h_2 \geq 0$ *on* $(0,1)$*;*
(ii) $h_1' \geq 0$, $h_2' \geq 0$ *on* $(0,1)$*;*
(iii) The elements in the second column of the matrix $\left[I - H(T-[T])\ C^{[T]}\right]^{-1}$ *are non-negative.*

Proof. It is obvious from the expression of the unique solution given in Theorem 3.1 [20]:

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^n\, V_0,\ t \in [n, n+1),\ n \in \mathbb{Z}^+,$$

where $V_0 = \left[I - H(T-[T])\ C^{[T]}\right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. □

Theorem 2. *Under the hypotheses of Theorem 1, and replacing Condition (iii) by:*

($\widetilde{iii}$) The elements in the second column of the matrix $\left[I - H(T-[T])\ C^{[T]}\right]^{-1}$ *are non-positive,*

the unique solution of problem (2) (with $s = 0$ or $s = T$) is non-positive.

Remark 1. *According to Lemmas A2–A6, conditions (i) and (ii) in Theorem 1 are fulfilled if the following conditions hold:*

- *If $b = 0$, $a \neq 0$:*
  $d \leq 0$ and one of the following hypotheses are satisfied:
  - *$c = 0$; or*
  - *$c \neq 0$, $a + c \leq 0$; or*
  - *$c \neq 0$, $a + c > 0$, $\frac{(a+c)e^{-a} - c}{a} \geq 0$;*
- *If $a = b = 0$:*
  $d \leq 0$ and $c \leq 1$;
- *If $b \neq 0$, $a^2 = 4b$:*
  $b + d \leq 0$, and $\frac{a}{2} + c \leq 1$;

- *If* $b \neq 0$, $a^2 > 4b$:
  $b + d \leq 0$, *and* $a + 2c \leq \sqrt{a^2 - 4b}\left(1 + \frac{2}{e^{\sqrt{a^2-4b}} - 1}\right)$;
- *If* $b \neq 0$, $a^2 < 4b$:
  One of the following conditions holds:
  * $b + d = 0$; *or*
  * $b + d < 0$ *and* $\sqrt{b - \frac{a^2}{4}} \leq \pi$,

  and one of the following conditions holds:
  * $\sqrt{b - \frac{a^2}{4}} \leq \frac{\pi}{2}$ *and* $\frac{a}{2} + c \leq 0$; *or*
  * $\sqrt{b - \frac{a^2}{4}} < \frac{\pi}{2}$ *and* $0 < \frac{a}{2} + c \leq \sqrt{b - \frac{a^2}{4}} \cot\left(\sqrt{b - \frac{a^2}{4}}\right)$; *or*
  * $\frac{\pi}{2} < \sqrt{b - \frac{a^2}{4}} < \pi$, *and* $\frac{a}{2} + c \leq \sqrt{b - \frac{a^2}{4}} \cot\left(\sqrt{b - \frac{a^2}{4}}\right) < 0$.

Theorem 3. *Theorems 1 and 2 are still valid, if we replace condition (ii) by the more general condition:*

$(\widetilde{ii})$ $\quad h_1'(1) \geq 0$, $h_2'(1) \geq 0$.

Remark 2. *Taking into account Theorem 3, it is possible to improve the conditions given in Remark 1, in the last case. According to Lemma A6, conditions (i) and* $(\widetilde{ii})$ *are fulfilled for the case* $b \neq 0$, $a^2 < 4b$, *if:*
one of the following conditions holds:

- $b + d = 0$; *or*
- $b + d < 0$ *and* $\sqrt{b - \frac{a^2}{4}} \leq \pi$,

and one of the following conditions holds:

- $\sqrt{b - \frac{a^2}{4}} \leq \frac{\pi}{2}$ *and* $\frac{a}{2} + c \leq 0$; *or*
- $\sqrt{b - \frac{a^2}{4}} < \frac{\pi}{2}$ *and* $0 < \frac{a}{2} + c \leq \sqrt{b - \frac{a^2}{4}} \cot\left(\sqrt{b - \frac{a^2}{4}}\right)$; *or*
- $\frac{\pi}{2} < \sqrt{b - \frac{a^2}{4}} < \pi$, *and* $\frac{a}{2} + c \leq \sqrt{b - \frac{a^2}{4}} \cot\left(\sqrt{b - \frac{a^2}{4}}\right) < 0$; *or*
- $\sqrt{b - \frac{a^2}{4}} \in (\pi, \frac{3\pi}{2})$, *and* $\frac{a}{2} + c > \sqrt{b - \frac{a^2}{4}} \cot\left(\sqrt{b - \frac{a^2}{4}}\right) > 0$, *provided that* $\mathcal{B} \geq 0$; *or*
- $\sqrt{b - \frac{a^2}{4}} = \frac{3\pi}{2}$, *and* $\frac{a}{2} + c > 0$, *provided that* $\mathcal{B} \geq 0$.

In the last two options, we consider:

$$\mathcal{B} = h_2(s_1) = e^{-\frac{a}{2}s_1}\left\{\frac{c}{b}\cos Rs_1 + \frac{1 + \frac{ac}{2b}}{R}\sin Rs_1\right\} - \frac{c}{b},$$

where,

$$R = \sqrt{b - \frac{a^2}{4}}, \text{ and } s_1 = \frac{1}{R}\left(\arctan\left(\frac{R}{\frac{a}{2} + c}\right) + \pi\right).$$

Note that:

$$\mathcal{B} = -e^{-\frac{a}{2}s_1}\left\{\frac{c}{b}\cos\left(\arctan\frac{R}{\frac{a}{2} + c}\right) + \frac{1 + \frac{ac}{2b}}{R}\sin\left(\arctan\frac{R}{\frac{a}{2} + c}\right)\right\} - \frac{c}{b}.$$

In Figure 1, we illustrate how the last cases included the consideration of different situations not covered by the estimates in Remark 1. Indeed, taking $R = \frac{3\pi}{2}$, $a = 7$, $b = R^2 + \frac{a^2}{4}$, *and* $c = -2.5$, *it is satisfied that* $b \neq 0$, $a^2 < 4b$, $\frac{a}{2} + c > 0$, *and* $\mathcal{B} \geq 0$, *and function* h_2 *is non-negative on* $(0, 1)$, *such that* $h_2'(1) > 0$.

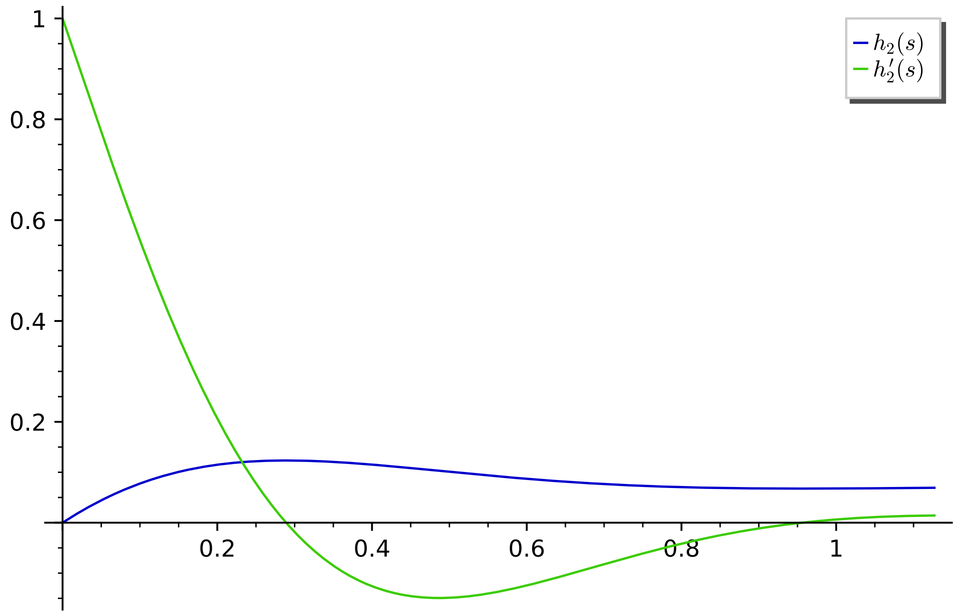

Figure 1. Graph of the functions h_2 and h_2' for $R = \frac{3\pi}{2}$, $a = 7$, $b = R^2 + \frac{a^2}{4}$, and $c = -2.5$.

Lemma 1. *Condition (iii) in Theorem 1 is fulfilled if the following conditions hold:*

$$\frac{\begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\det\left(I - H(T-[T])\, C^{[T]}\right)} \geq 0, \tag{5}$$

$$\frac{1 - \begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\det\left(I - H(T-[T])\, C^{[T]}\right)} \geq 0. \tag{6}$$

Proof. We use that, for $A = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$,

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} r_4 & -r_2 \\ -r_3 & r_1 \end{pmatrix},$$

in consequence, taking $A = (I - H(T-[T])C^{[T]})$, the first row of A is:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]},$$

and the second row of A is:

$$\begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} h_1'(T-[T]) & h_2'(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]}.$$

Thus, the elements in the second column of A^{-1} are given by:

$$\frac{\begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\det\left(I - H(T-[T])\ C^{[T]}\right)}$$

and

$$\frac{1 - \begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\det\left(I - H(T-[T])\ C^{[T]}\right)}.$$

□

Remark 3. *If* $\det\left(I - H(T-[T])C^{[T]}\right) > 0$*, conditions (5) and (6) hold if:*

$$\begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \tag{7}$$

has the first component less than or equal to 1 *and the second component is non-negative. However, in the context of Theorem 1, due to the validity of i) and ii), (or, more generally, i) and* $\widetilde{ii}$*)), it is clear that the part above in (5) is non-negative, thus, if* $\det\left(I - H(T-[T])C^{[T]}\right) > 0$*, and the first component in (7) is less than or equal to* 1*, then iii) is satisfied.*

Similarly, if $\det\left(I - H(T-[T])C^{[T]}\right) < 0$*, conditions (5) and (6) hold if (7) has the first component greater than or equal to* 1 *and the second component non-positive (null, by the restrictions).*

Lemma 2. *The reverse inequalities in Lemma 1 imply the validity of* $(\widetilde{iii})$ *in Theorem 2.*

Remark 4. *In relation with Theorem 2, under the hypotheses (i) and* $(\widetilde{ii})$*, it is possible to assure that, if* $\det\left(I - H(T-[T])C^{[T]}\right) < 0$*, then the opposite to (5) holds. Assuming also that the first component in (7) is less than or equal to* 1*, then the opposite to (6) holds. Thus, by Lemma 2, these two conditions imply the validity of* $(\widetilde{iii})$*.*

On the other hand, if $\det\left(I - H(T-[T])C^{[T]}\right) > 0$*,* $(\widetilde{iii})$ *holds if the first component in (7) is greater than or equal to* 1 *and the second component in (7) is null.*

Next, we study the sign of the solution to the problem (2), for $0 < s < T$ with $s \in \mathbb{Z}^+$.

Theorem 4. *Assume that condition (4) holds. For* $0 < s < T$*,* $s \in \mathbb{Z}^+$*, the unique solution to problem (2) is non-negative if the following conditions hold: (i) in Theorem 1,* $(\widetilde{ii})$ *in Theorem 3 (see Remarks 1 and 2), and*

$(\widehat{iii})$ *The elements in the second column of the matrix:*

$$\left[I - H(T-[T])\ C^{[T]}\right]^{-1} H(T-[T])\ C^{[T]-s}$$

are non-negative.

Proof. It is deduced from the expression of the unique solution to (2) given in Theorem A4 (Theorem 3.1 [20]):

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^n\, V_0,\ t \in [n, n+1),\ n \in \{0, 1, \ldots, s-1\},$$

and

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^{n-s} \left[C^s V_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right], \; t \in [n, n+1), \; n \geq s,$$

where

$$V_0 = \left[I - H(T-[T]) \; C^{[T]} \right]^{-1} H(T-[T]) \; C^{[T]-s} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that $V_0 \geq 0$ (non-negative components) implies that $C^s V_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0$, that is, this vector has also non-negative components. □

Lemma 3. *Assume one of the following hypotheses:*

- *Conditions (i), (ii) in Theorem 1, that is,* h_1, h_2, h_1' *and* h_2' *are non-negative on* $(0,1)$*; or more generally*
- *Condition (i) in Theorem 1, and* $h_1'(1) \geq 0$, $h_2'(1) \geq 0$, $h_1'(T-[T]) \geq 0$, $h_2'(T-[T]) \geq 0$.

Then, condition $(\widehat{iii})$ *in Theorem 4 is satisfied if* $\left[I - H(T-[T]) \; C^{[T]} \right]^{-1}$ *has non-negative components. In particular, this last condition is valid if* $\det \left(I - H(T-[T]) \; C^{[T]} \right) > 0$ *and the following numbers are non-negative:*

$$1 - \begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \geq 0,$$

$$1 - \begin{pmatrix} h_1'(T-[T]) & h_2'(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0.$$

Proof. Under these particular conditions, and using the same procedure as in the proof of Lemma 1, we prove that $[I - H(T-[T])C^{[T]}]^{-1}$ has non-negative components, since, obviously,

$$\begin{pmatrix} h_1(T-[T]) & h_2(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0,$$

and

$$\begin{pmatrix} h_1'(T-[T]) & h_2'(T-[T]) \end{pmatrix} \begin{pmatrix} c_1 & c_2 \\ c_1' & c_2' \end{pmatrix}^{[T]} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \geq 0.$$

□

Theorem 5. *Assume that condition (4) holds. For* $0 < s < T$, $s \in \mathbb{Z}^+$*, the unique solution to problem (2) is non-positive if the following conditions hold: (i) in Theorem 1,* $(\widetilde{ii})$ *in Theorem 3 (see Remarks 1 and 2), and*

(iv) The elements in V_0*, the second column of the matrix:*

$$\left[I - H(T-[T]) \; C^{[T]} \right]^{-1} H(T-[T]) \; C^{[T]-s},$$

are non-positive, and the elements of $C^s V_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *are non-positive.*

Remark 5. *The obvious changes in Lemma 3 lead to* $V_0 \leq 0$*, which joint to Conditions (i),* $(\widetilde{ii})$ *implies that* $K(t,s) \leq 0$ *for* $0 < s < T$, $s \in \mathbb{Z}^+$*, and* $t \in [n, n+1)$, $n \in \{0, 1, \ldots, s-1\}$*. To obtain the non-positiveness of* $K(t,s)$ *on* $[n, n+1)$*, for* $n \geq s$*, we have to guarantee that* $C^s V_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leq 0$.

Next, we give some extensions of Theorems 1, and 2 by virtue of the following Lemma.

Lemma 4. *Suppose that $h_1(1) > 0$, and that $h_2 > 0$ on $(0,1)$. Define the set:*

$$\mathcal{S} := \{(x,y) \in \mathbb{R}^2 \, : \, x \geq 0, \, y \geq -Mx\},$$

with

$$M = \inf_{z \in (0,1)} \frac{h_1(z)}{h_2(z)}.$$

If $(x,y) \in \mathcal{S}$, then $h_1(z)x + h_2(z)y \geq 0$ for every $z \in [0,1]$.

Proof. Let $(x,y) \in \mathcal{S}$. Then:

- If $z = 0$, by the properties of h_1 and h_2, $h_1(0)x + h_2(0)y = x \geq 0$ by the definition of $\mathcal{S}$;
- On the other hand, for $z \in (0,1)$, $h_2(z) > 0$ and, since $x \geq 0$,

$$y \geq -\inf_{\xi \in (0,1)} \frac{h_1(\xi)}{h_2(\xi)} x \geq -\frac{h_1(z)}{h_2(z)} x,$$

  thus the non-negativity of $h_1(z)x + h_2(z)y$ is derived;
- Finally, if $z = 1$, we have $h_1(1)x + h_2(1)y$ and we distinguish two cases. If $h_2(1) = 0$, it is reduced to $h_1(1)x$, which is non-negative, and if $h_2(1) > 0$, then, by the continuity of h_1 and h_2,

$$M = \inf_{\xi \in (0,1)} \frac{h_1(\xi)}{h_2(\xi)} = \inf_{\xi \in (0,1]} \frac{h_1(\xi)}{h_2(\xi)},$$

  and we conclude.

□

Theorem 6. *Assume that condition (4) holds. For $s = 0$ or $s = T$, the unique solution to problem (2) is non-negative if the following conditions hold:*

(I) $h_1(1) > 0$, *and* $h_2 > 0$ *on* $(0,1)$;
(II) *The vector V_0 given by:*

$$V_0 = \left[I - H(T - [T]) \; C^{[T]} \right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

satisfies that $C^k \, V_0 \in \mathcal{S}$ for every $k = 0, 1, \ldots, [T]$.

Theorem 7. *Assume that condition (4) holds. For $s = 0$ or $s = T$, the unique solution to problem (2) is non-positive if the following conditions hold:*

(I) $h_1(1) > 0$, *and* $h_2 > 0$ *on* $(0,1)$;
(II)* *The vector V_0 given by (8) satisfies that $(-1)C^k \, V_0 \in \mathcal{S}$ for every $k = 0, 1, \ldots, [T]$.*

Similar considerations can be made for Theorems 4 and 5. We omit them since they are not required for the study of the sign of the nonhomogeneus Equation (1).

Theorem 8. *Assume that condition (4) holds. If $0 < s < T$, $s \in (n, n+1)$, for some $n \in \mathbb{Z}^+$, then the unique solution to problem (2) is non-negative if the following conditions hold:*

(I) $h_1(1) > 0$, *and* $h_2 > 0$ *on* $(0,1)$;
(III) *The function g given by (3) is non-negative on* $(0,1)$;

(IV) *The vector V_0 given, for $T < n+1$, by:*

$$V_0 = \left[I - H(T-[T])\ C^{[T]}\right]^{-1} \begin{pmatrix} g(T-s) \\ g'(T-s) \end{pmatrix},$$

and, for $n+1 \leq T$, by

$$V_0 = \left[I - H(T-[T])\ C^{[T]}\right]^{-1} H(T-[T])\ C^{[T]-n-1} \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix},$$

satisfies that $C^k V_0 \in \mathcal{S}$, for every $k = 0, 1, \ldots, [T]$;

(V) *If $T \geq n+1$, assume also that the vector:*

$$V_1 = \left[C^{n+1} V_0 + \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix}\right], \qquad \text{(with } V_0 \text{ given in IV)},$$

satisfies that $C^k V_1 \in \mathcal{S}$, for every $k = 0, 1, \ldots, [T] - n - 1$.

Proof. In Theorem 3.1 [20] (see Theorem A4), it is proved that, under these assumptions, the expression of the unique solution to problem (2), for $0 < s < T$, $s \in (n, n+1)$, for some $n \in \mathbb{Z}^+$, is given by:

$$K(t,s) = \begin{pmatrix} h_1(t-k) & h_2(t-k) \end{pmatrix} C^k\, V_0,\ t \in [k, k+1) \cap [0, s),\ k \in \{0, 1, \ldots, n\},$$

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^n\, V_0 + g(t-s),\ t \in [s, n+1) \cap [0, T],$$

$$\begin{aligned} K(t,s) = {} & \begin{pmatrix} h_1(t-n-1-k) & h_2(t-n-1-k) \end{pmatrix} C^{n+1+k}\, V_0 \\ & + \begin{pmatrix} h_1(t-n-1-k) & h_2(t-n-1-k) \end{pmatrix} C^k \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix}, \end{aligned}$$

for $t \in [n+1+k, n+2+k) \cap [0, T]$, $k \in \mathbb{Z}^+$, where V_0 is given in IV) and g is given by (3).

Hence, the proof is concluded by using Lemma 4. □

Remark 6. *According to Lemmas A2–A6, Condition (I) in Theorems 6–8 is satisfied under the following circumstances:*

- $b = 0$, $a \neq 0$, $1 - \dfrac{d}{a} + \dfrac{d}{a^2}(1 - e^{-a}) > 0$, *and one of the following conditions hold:*
  - ∗ $c = 0$; *or*
  - ∗ $c \neq 0$, *and* $a + c \leq 0$; *or*
  - ∗ $c \neq 0$, $a + c > 0$, *and* $\frac{(a+c)e^{-a} - c}{a} \geq 0$; *or*
  - ∗ $c \neq 0$, $a + c > 0$, $\frac{(a+c)e^{-a} - c}{a} < 0$, *and* $\dfrac{1}{a}\left(1 - e^{-a} - c + \dfrac{c}{a}(1 - e^{-a})\right) \geq 0$.
- $b = 0$, $a = 0$, $1 - \dfrac{d}{2} > 0$, *and one of the following conditions hold:*
  - ∗ $c \leq 1$; *or*
  - ∗ $c > 1$, *and* $c \leq 2$.
- $b \neq 0$, $a^2 = 4b$, $\left(1 + \dfrac{d}{b}\right)\left(1 + \dfrac{a}{2}\right) e^{-\frac{a}{2}} - \dfrac{d}{b} > 0$, *and one of the following conditions hold:*

  * $\frac{a}{2}+c \le 1$; *or*
  * $\frac{a}{2}+c > 1$, *and* $e^{-\frac{a}{2}}\left[\frac{c}{b}\left(1+\frac{a}{2}\right)+s\right]-\frac{c}{b} \ge 0.$
- $b \neq 0,\ a^2 > 4b,\ \left(1+\frac{d}{b}\right)\frac{\beta e^{\alpha}-\alpha e^{\beta}}{\beta-\alpha}-\frac{d}{b} > 0$, *and one of the following conditions hold:*
  * $a+2c \le \sqrt{a^2-4b}\left(1+\frac{2}{e^{\sqrt{a^2-4b}}-1}\right)$; *or*
  * $a+2c > \sqrt{a^2-4b}\left(1+\frac{2}{e^{\sqrt{a^2-4b}}-1}\right)$, *and* $\frac{\left(\frac{\beta c}{b}-1\right)e^{\alpha}+\left(1-\frac{\alpha c}{b}\right)e^{\beta}}{\beta-\alpha}-\frac{c}{b} \ge 0.$
- $b \neq 0,\ a^2 < 4b,\ \left(1+\frac{d}{b}\right)e^{-\frac{a}{2}}\left\{\cos R+\frac{a}{2R}\sin R\right\}-\frac{d}{b} > 0$, *where* $R := \sqrt{b-\frac{a^2}{4}}$, *and one of the following conditions hold:*
  * $R \le \frac{\pi}{2}$ *and* $\frac{a}{2}+c \le 0$; *or*
  * $R < \frac{\pi}{2}$, *and* $0 < \frac{a}{2}+c \le R\cot(R)$; *or*
  * $\frac{\pi}{2} < R < \pi$, *and* $\frac{a}{2}+c \le R\cot(R) < 0$; *or*
  * $e^{-\frac{a}{2}}\left\{\frac{c}{b}\cos R+\frac{1+\frac{ac}{2b}}{R}\sin R\right\}-\frac{c}{b} \ge 0$, *and one of the following restrictions hold:*
    ⋆ $R = \frac{\pi}{2}$ *and* $\frac{a}{2}+c > 0$; *or*
    ⋆ $R < \frac{\pi}{2}$ *and* $\frac{a}{2}+c > R\cot(R)$; *or*
    ⋆ $R \in (\frac{\pi}{2}, \pi]$, *and* $\frac{a}{2}+c > 0$; *or*
    ⋆ $R \in (\pi, \frac{3\pi}{2})$, *and* $0 < \frac{a}{2}+c \le R\cot(R)$; *or*
    ⋆ $\frac{3\pi}{2} \ge R \ge \pi$, *and* $\frac{a}{2}+c < 0$; *or*
    ⋆ $\frac{\pi}{2} < R < \pi$, *and* $0 > \frac{a}{2}+c > R\cot(R)$.
  * $e^{-\frac{a}{2}s_1}\left\{\frac{c}{b}\cos Rs_1+\frac{1+\frac{ac}{2b}}{R}\sin Rs_1\right\}-\frac{c}{b} > 0$ *and one of the following restrictions hold:*
    ⋆ $R \in (\pi, \frac{3\pi}{2})$, *and* $\frac{a}{2}+c > R\cot(R) > 0$; *or*
    ⋆ $R = \frac{3\pi}{2}$, *and* $\frac{a}{2}+c > 0$.
  * $\frac{a}{2}+c = 0$, *and* $R \le \frac{\pi}{2}$.

Remark 7. *According to Lemmas A7–A11, Condition (III) in Theorem 8 is satisfied if one of the following conditions holds:*

- $b = 0$;
- $b \neq 0,\ a^2 \ge 4b$;
- $b \neq 0,\ a^2 < 4b$, *and* $R \le \pi$.

Remark 8. *If* $T \in (0,1)$, *Conditions IV) and V) in Theorem 8 are reduced to* $V_0 \in \mathcal{S}$. *If* $T \ge 1$, *then several successive iterations by C of the vectors in IV) and V) should belong to the set* $\mathcal{S}$.

To show the first iteration, suppose that $h_1(1) > 0$, $h_2 > 0$ *on* $(0,1)$, *and* $h_2'(1)+Mh_2(1) > 0$. *Let* $V=(x,y) \in \mathcal{S}$. *Then V is such that* $CV \in \mathcal{S}$ *if and only if:*

$$h_1(1)x+h_2(1)y \ge 0, \qquad h_1'(1)x+h_2'(1)y \ge -M(h_1(1)x+h_2(1)y). \tag{9}$$

Since $(x,y) \in \mathcal{S}$, $x \ge 0$ *and* $y \ge -\inf_{z\in(0,1)}\frac{h_1(z)}{h_2(z)}x$. *If* $h_2(1)=0$, *then* $h_1(1)x \ge 0$ *is trivially valid by hypotheses and, if* $h_2(1) > 0$,

$$y \ge -\inf_{z\in(0,1)}\frac{h_1(z)}{h_2(z)}x = -\inf_{z\in(0,1]}\frac{h_1(z)}{h_2(z)}x \ge -\frac{h_1(1)}{h_2(1)}x,$$

so that the first inequality in (9) holds. On the other hand, if

$$y \geq -\frac{h_1'(1) + Mh_1(1)}{h_2'(1) + Mh_2(1)}x,$$

the second inequality in (9) holds. Hence, the interesting set for one iteration is:

$$\mathcal{S}_1 := \left\{ (x,y) \in \mathbb{R}^2 \; : \; x \geq 0, \, y \geq -Mx, \, y \geq -\frac{h_1'(1) + Mh_1(1)}{h_2'(1) + Mh_2(1)}x \right\}.$$

If we define the (finite) sequence (assuming that the denominators are positive):

$$M_0 := M, \qquad M_n = \frac{h_1'(1) + M_{n-1}h_1(1)}{h_2'(1) + M_{n-1}h_2(1)}, \; n = 1, \ldots, l,$$

the interesting set for iterations until order l is:

$$\mathcal{S}_l := \left\{ (x,y) \in \mathbb{R}^2 \; : \; x \geq 0, \, y \geq -\left(\min_{k=0,1,\ldots,l} M_k \right) x \right\}.$$

If we are interested in analyzing the sign of the solution to the second-order functional-differential equation with piecewise constant arguments (1), which exists and is unique under the appropriate hypothesis, we need to determine the sign of $K(t,0)$ and $K(t,s)$ on $[0,T] \times [0,T]$ except at most in a set of measure zero. This is a consequence of the fact that the expression of $K(t,s)$, the solution to problem (2), represents the Green's function for the solution to problem (1), as established in Theorem 3.2 [20].

Theorem 9. *Suppose that hypothesis (4) holds. If $\sigma \in \Lambda$ is non-negative on J, $\lambda \geq 0$, conditions (I), (II) in Theorem 6 hold and conditions (III), (IV), and (V) in Theorem 8 hold [it is assumed that (IV) and (V) will be fulfilled for all $0 < s < T$, with $s \in (n, n+1)$, for some $n \in \mathbb{Z}^+$], then the unique solution to problem (1) is non-negative on J.*

Proof. This result is obtained by applying Theorems 6, and 8, and using that the unique solution to problem (1) is, according to Theorem A5 (Theorem 3.2 [20]), given by:

$$x(t) = \int_0^T K(t,s)\, \sigma(s)\, ds + \lambda K(t,0), \; t \in J,$$

where, for all $s \in J$, $K(\cdot, s)$ is the unique solution to problem (2). □

Theorem 10. *Suppose that the hypothesis (4) holds. Assume also that the conditions (I), (II) in Theorem 6 hold and conditions (III), (IV), and (V) in Theorem 8 hold [it is assumed (IV) and (V) are to be fulfilled for all $0 < s < T$, with $s \in (n, n+1)$, for some $n \in \mathbb{Z}^+$], if $\sigma \in \Lambda$ is non-positive on J and $\lambda \leq 0$, then the unique solution to problem (1) is non-positive on J.*

Remark 9. *Note that we only need to impose Condition (V) in Theorems 9 and 10 if $T > n + 1$, since the value of $K(T, s)$ is not required.*

3. Examples

Finally, we present some examples to illustrate the main results.

Example 1. *Consider $b = 0$, $a \neq 0$, and $0 < T < 1$. We check that conditions in Theorem 1 hold. Indeed, according to Remark 1, conditions (i), (ii) are true for $d \leq 0$ and one of the following hypotheses:*

- $c = 0$; *or*
- $c \neq 0$, $a + c \leq 0$; *or*
- $c \neq 0$, $a + c > 0$, $\frac{(a+c)e^{-a} - c}{a} \geq 0$.

For condition (iii), using that $[T] = 0$, then the matrix:

$$\begin{aligned} I - H(T - [T])\ C^{[T]} &= I - H(T - [T]) = I - H(T) \\ &= I - \begin{pmatrix} 1 - \frac{d}{a}T + \frac{d}{a^2}(1 - e^{-aT}) & \frac{1}{a}(1 - e^{-aT} - cT + \frac{c}{a}(1 - e^{-aT}) \\ \frac{d}{a}(e^{-aT} - 1) & \frac{1}{a}(ae^{-aT} - c + ce^{-aT}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{d}{a}T - \frac{d}{a^2}(1 - e^{-aT}) & -\frac{1}{a}(1 - e^{-aT} - cT + \frac{c}{a}(1 - e^{-aT})) \\ -\frac{d}{a}(e^{-aT} - 1) & 1 - \frac{1}{a}(ae^{-aT} - c + ce^{-aT}) \end{pmatrix} \end{aligned}$$

Note that $\det(I - H(T)) = \frac{dT}{a}(1 - e^{-aT}) \neq 0$ *if and only if $d \neq 0$, due to $a \neq 0$.*

Suppose that $d \neq 0$. According to Lemma 1, for the validity of (5) and (6), we only have to check that:

$$\frac{h_2(T)}{\det(I - H(T))} \geq 0, \tag{10}$$

and

$$\frac{1 - h_1(T)}{\det(I - H(T))} \geq 0. \tag{11}$$

Indeed, we have to impose that:

$$\begin{aligned} \frac{h_2(T)}{\det(I - H(T))} &= \frac{\frac{1}{a}((1 - e^{-aT})(1 + \frac{c}{a}) - cT)}{\frac{dT}{a}(1 - e^{-aT})} \\ &= \frac{(1 - e^{-aT})(1 + \frac{c}{a}) - cT}{dT(1 - e^{-aT})} = \frac{(1 - e^{-aT})(a + c) - acT}{daT(1 - e^{-aT})} \geq 0, \end{aligned}$$

and

$$\frac{1 - h_1(T)}{\det(I - H(T))} = \frac{\frac{d}{a^2}(aT - 1 + e^{-aT})}{\frac{dT}{a}(1 - e^{-aT})} = \frac{aT - 1 + e^{-aT}}{aT(1 - e^{-aT})} \geq 0. \tag{12}$$

We take into account that function $v(z) = z - (1 - e^{-z})$ is positive for $z \neq 0$, because $v(0) = 0$ and $v'(z) = 1 - e^{-z} > 0$ if $z > 0$ and $v'(z) < 0$ if $z < 0$. Then $v(aT) > 0$, since $a \neq 0$, and hence the condition we must assume for the validity of (12) is:

$$aT(1 - e^{-aT}) > 0,$$

which is trivially true independently of the value of $a \neq 0$. Thus, condition (12) is trivially satisfied.

On the other hand, for the validity of (10), we must assume that:

$$\frac{(1 - e^{-aT})(a + c) - acT}{d} \geq 0. \tag{13}$$

This proves that, under condition (13), hypotheses (5) and (6) hold. Therefore, after analyzing the cases that lead to incompatible conditions, it is possible to affirm that, if $a < 0$, $b = 0$, $c = 0$, $d < 0$, and $0 < T < 1$, then the unique solution to problem (2) is non-negative, for $s = 0$ or $s = T$.

Example 2. *Similarly to Example 1, consider $b = 0$, $a \neq 0$, and $0 < T < 1$. We impose the following restrictions on the constants $c = 0$, and $a > d > 0$.*

Condition (4) is satisfied due to $d \neq 0$ (see Example 1). Condition (I) is fulfilled (see Remark 6) since $1 - \dfrac{d}{a} + \dfrac{d}{a^2}(1 - e^{-a}) > 0$ is valid by the choice of the constants. On the other hand, condition (III) is trivially satisfied by virtue of Remark 7. Since $T \in (0,1)$ and the choice of the constants,

$$\left[I - H(T - [T])\, C^{[T]}\right]^{-1} = \begin{pmatrix} \frac{d}{a}T - \frac{d}{a^2}(1 - e^{-aT}) & -\frac{1}{a}(1 - e^{-aT}) \\ -\frac{d}{a}(e^{-aT} - 1) & 1 - e^{-aT} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a}{dT} & \frac{1}{dT} \\ -\frac{1}{T} & \frac{aT-1+e^{-aT}}{aT(1-e^{-aT})} \end{pmatrix}.$$

Note that it is not true that all the elements in the matrix $\left[I - H(T - [T])\; C^{[T]}\right]^{-1}$ are non-negative, since its element $(2,1)$ is negative: $-\frac{1}{T} < 0$. Next, we check condition II). Consider V_0 given by:

$$V_0 = \left[I - H(T - [T])\; C^{[T]}\right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{dT} \\ \frac{aT-1+e^{-aT}}{aT(1-e^{-aT})} \end{pmatrix},$$

and check that $V_0 \in \mathcal{S}$. In this case,

$$\mathcal{S} := \{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0,\, y \geq -Mx\},$$

with

$$M = \inf_{z\in(0,1)} \frac{h_1(z)}{h_2(z)} = \inf_{z\in(0,1)} \frac{a - dz + \frac{d}{a}(1 - e^{-az})}{(1 - e^{-az})} = \inf_{z\in(0,1)} \frac{a - dz}{1 - e^{-az}} + \frac{d}{a}.$$

Let $\varphi(z) := \frac{a-dz}{1-e^{-az}}$, $z \in (0,1)$. The sign of φ' coincides with the sign of function ψ, defined on $[0,1]$ by $\psi(z) = d - a^2 + adz - de^{az}$. The derivative $\psi'(z) = da(1 - e^{az})$ is negative and $\psi(0) = -a^2 < 0$. Hence $\psi < 0$ on $[0,1]$ and $\varphi' < 0$ on $(0,1)$. In consequence,

$$M = \frac{a - d}{1 - e^{-a}} + \frac{d}{a} > 0.$$

Hence, $V_0 \in \mathcal{S}$, since $\frac{1}{dT} > 0$ and

$$\frac{aT - 1 + e^{-aT}}{aT(1 - e^{-aT})} \geq -\left(\frac{a - d}{1 - e^{-a}} + \frac{d}{a}\right)\frac{1}{dT}.$$

Indeed, this inequality is equivalent to:

$$\frac{aT - 1 + e^{-aT}}{1 - e^{-aT}} \geq -\left(\frac{a - d}{1 - e^{-a}} + \frac{d}{a}\right)\frac{a}{d} = -\left(\frac{a(a - d)}{d(1 - e^{-a})} + 1\right),$$

and also to:

$$\frac{-aT}{1 - e^{-aT}} = \frac{1 - aT - e^{-aT}}{1 - e^{-aT}} - 1 \leq \frac{a(a - d)}{d(1 - e^{-a})},$$

which is trivially valid since the left-hand side is negative and the right-hand side is positive.

On the other hand, we check IV), proving that V_0 given by:

$$\begin{aligned} V_0 &= \left[I - H(T-[T])\ C^{[T]}\right]^{-1} \begin{pmatrix} g(T-s) \\ g'(T-s) \end{pmatrix} \\ &= \begin{pmatrix} \frac{a}{dT} & \frac{1}{dT} \\ -\frac{1}{T} & \frac{aT-1+e^{-aT}}{aT(1-e^{-aT})} \end{pmatrix} \begin{pmatrix} \frac{1-e^{-a(T-s)}}{a} \\ e^{-a(T-s)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{dT} \\ -\frac{1}{aT}\left(1-e^{-a(T-s)} + \frac{1-aT-e^{-aT}}{1-e^{-aT}} e^{-a(T-s)}\right) \end{pmatrix} \end{aligned}$$

belongs to $\mathcal{S}$, for every $s \in (0,T)$. Indeed, $V_0 \in \mathcal{S}$, since $\frac{1}{dT} > 0$ and

$$-\frac{1}{aT}\left(1-e^{-a(T-s)} + \frac{1-aT-e^{-aT}}{1-e^{-aT}} e^{-a(T-s)}\right) \geq -\left(\frac{a-d}{1-e^{-a}} + \frac{d}{a}\right)\frac{1}{dT},$$

which is rewritten as:

$$1-e^{-a(T-s)} + \frac{1-aT-e^{-aT}}{1-e^{-aT}} e^{-a(T-s)} \leq \left(\frac{a-d}{1-e^{-a}} + \frac{d}{a}\right)\frac{a}{d} = \frac{a(a-d)}{d(1-e^{-a})} + 1,$$

or

$$e^{-a(T-s)}\left[\frac{1-aT-e^{-aT}}{1-e^{-aT}} - 1\right] \leq \frac{a(a-d)}{d(1-e^{-a})},$$

that is,

$$\frac{-aTe^{-aT}}{1-e^{-aT}} e^{as} \leq \frac{a(a-d)}{d(1-e^{-a})},$$

which is trivially valid, since the left-hand side is negative and the right-hand side is positive.

Therefore, if $\sigma \in \Lambda$ is non-negative on J, and $\lambda \geq 0$, then the unique solution to problem (1) is non-negative on J.

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Appendix A. Some Preliminary Results

For the proof of the results in this appendix, see [20].

Note that, for $\sigma \equiv 0$ and $\lambda = 0$, the trivial function satisfies the conditions in (1).

Lemma A1 (c.f. Lemma 1, [20]). *The solutions for the initial value problem:*

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = 0,\ t \in [n, n+1), \\ x(n^+) = v_n, \\ x'(n^+) = v'_n, \end{cases} \tag{A1}$$

where v_n, $v'_n \in \mathbb{R}$, are obtained in terms of a and b as:

$$x(t) = v_n\, h_1(t-n) + v'_n\, h_2(t-n),\ t \in [n, n+1), \tag{A2}$$

where h_1 and h_2 are defined at the starting of Section 2.

As stated in [20], functions h_1 and h_2 satisfy the following interesting properties:

$$h_1(0) = 1,\ h_2(0) = 0,\ h'_1(0) = 0,\ h'_2(0) = 1.$$

Theorem A1 (c.f. Theorem 2.1, [20]). *The initial value problem:*

$$\begin{cases} v''(t) + av'(t) + bv(t) + cv'([t]) + dv([t]) = 0,\ t \in [0, +\infty), \\ v(0) = v_0, \\ v'(0) = v'_0, \end{cases} \tag{A3}$$

for v_0, $v'_0 \in \mathbb{R}$, has the solution:

$$v(t) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C'_1 & C'_2 \end{pmatrix}^n \begin{pmatrix} v_0 \\ v'_0 \end{pmatrix},\ t \in [n, n+1), \tag{A4}$$

where $n \in \mathbb{Z}^+$,

$$C_1 = h_1(1),\ \ C_2 = h_2(1),$$
$$C'_1 = h'_1(1),\ \ C'_2 = h'_2(1),$$

and h_1, h_2 are expressed at the starting of Section 2.

Consider $V_0 = \begin{pmatrix} v_0 \\ v'_0 \end{pmatrix}$, and I the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Theorem A2 (c.f. Theorem 2.2, [20]). *For $T > 0$, the solution $v(t)$ of the initial value problem (A3) satisfies*

$$v(T) = v_0,\ \ v'(T) = v'_0$$

if and only if:

$$\left[I - H(T-[T])C^{[T]} \right] V_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{A5}$$

Moreover, if (4) holds, then $v_0 = v'_0 = 0$ and we obtain the trivial solution. If,

$$det\left(I - H(T-[T])C^{[T]} \right) = 0, \tag{A6}$$

then there exist nontrivial solutions of (A3) with $v(T) = v_0$ and $v'(T) = v'_0$.

Corollary A1 (c.f. Corollary 1, [20]). *Problem (1) with $\sigma \equiv 0$ and $\lambda = 0$, that is,*

$$\begin{cases} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) = 0, \; t \in J, \\ x(0) = x(T), \\ x'(0) = x'(T), \end{cases}$$

has nontrivial solutions if and only if (A6) is valid. Such solutions can be obtained from expression (A4), taking the initial conditions v_0, v_0' satisfying (A5).

Note that a necessary (non sufficient) condition for the validity of (4) is $b + d \neq 0$.

Theorem A3 (c.f. Theorem 2.3, [20]). *If $b + d = 0$, then (A6) holds.*

If problem (2) has a unique solution, for $s \in J$ fixed, we denote it by $K(t,s)$, the value of the solution for problem (2) at the point t.

Recall that function g is defined in (3).

Theorem A4 (c.f. Theorem 3.1, [20]). *For $s \in J$ fixed, problem (2) has a unique solution if and only if condition (4) holds, that is, the matrix:*

$$I - H(T - [T])\, C^{[T]} \text{ is invertible.} \tag{A7}$$

In such a case, the solution is given in J by the following expression:

- *If $s = 0$ or $s = T$,*

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^n\, V_0, \; t \in [n, n+1), \; n \in \mathbb{Z}^+,$$

  where,

$$V_0 = \left[I - H(T - [T])\; C^{[T]} \right]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- *If $0 < s < T$, $s \in \mathbb{Z}^+$,*

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^n\, V_0, \; t \in [n, n+1), \; n \in \{0, 1, \ldots, s-1\},$$

  and

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^{n-s} \left[C^s V_0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right], \; t \in [n, n+1), \; n \geq s,$$

  where,

$$V_0 = \left[I - H(T - [T])\; C^{[T]} \right]^{-1} H(T - [T])\; C^{[T]-s} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- *If $0 < s < T$, $s \in (n, n+1)$, for some $n \in \mathbb{Z}^+$,*

$$K(t,s) = \begin{pmatrix} h_1(t-k) & h_2(t-k) \end{pmatrix} C^k\, V_0, \; t \in [k, k+1) \cap [0, s), \; k \in \{0, 1, \ldots, n\},$$

$$K(t,s) = \begin{pmatrix} h_1(t-n) & h_2(t-n) \end{pmatrix} C^n\, V_0 + g(t-s), \; t \in [s, n+1) \cap [0, T],$$

$$K(t,s) = \begin{pmatrix} h_1(t-n-1-k) & h_2(t-n-1-k) \end{pmatrix} C^{n+1+k}\, V_0 + \\ + \begin{pmatrix} h_1(t-n-1-k) & h_2(t-n-1-k) \end{pmatrix} C^k \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix},$$

for $t \in [n+1+k, n+2+k) \cap [0,T]$, $k \in \mathbb{Z}^+$,
where, for $T < n+1$,

$$V_0 = \left[I - H(T-[T])\; C^{[T]} \right]^{-1} \begin{pmatrix} g(T-s) \\ g'(T-s) \end{pmatrix},$$

and, for $n+1 \le T$,

$$V_0 = \left[I - H(T-[T])\; C^{[T]} \right]^{-1} H(T-[T])\; C^{[T]-n-1} \begin{pmatrix} g(n+1-s) \\ g'(n+1-s) \end{pmatrix}.$$

Theorem A5 (c.f. Theorem 3.2, [20])**.** *If hypothesis (4) holds, problem (1) has a unique solution, for all $\sigma \in \Lambda$ (see Definition 2) and $\lambda \in \mathbb{R}$, which can be obtained by the expression:*

$$x(t) = \int_0^T K(t,s)\,\sigma(s)\,ds + \lambda K(t,0),\; t \in J, \tag{A8}$$

where, for all $s \in J$, $K(\cdot, s)$ is the unique solution to (2).

Appendix B. Technical Results Concerning the Properties of Functions h_1 and h_2

Lemma A2. *If $b = 0$, $a \neq 0$, the following properties hold:*

- *If $d < 0$, then $h_1' > 0$ on $(0,1)$.*
- *If $d > 0$, then $h_1' < 0$ on $(0,1)$. In this case, if $h_1(1) \ge 0$, then $h_1 > 0$ on $(0,1)$.*
- *If $d = 0$, then $h_1' = 0$ on $(0,1)$.*
- *If $d \le 0$, then $h_1 > 0$ on $(0,1)$.*
- *$h_2' > 0$ on $(0,1)$, if one of the following conditions hold:*
    - ∗ *$c = 0$; or*
    - ∗ *$c \neq 0$, and $a + c \le 0$; or*
    - ∗ *$c \neq 0$, $a + c > 0$, and $\frac{(a+c)e^{-a} - c}{a} \ge 0$.*

  In any case, $h_2 \ge 0$ on $(0,1)$.
- *If $c \neq 0$, $a + c > 0$ and $\frac{(a+c)e^{-a} - c}{a} < 0$, then $h_2' > 0$ on $(0, -\frac{1}{a}\ln\frac{c}{a+c})$ and $h_2' < 0$ on $(-\frac{1}{a}\ln\frac{c}{a+c}, 1)$. In this case, if $h_2(1) \ge 0$, then $h_2 > 0$ on $(0,1)$.*

Proof. In the case $b = 0$, $a \neq 0$, by Lemma A1, $h_1(s) = 1 - \dfrac{d}{a}s + \dfrac{d}{a^2}(1 - e^{-as})$, hence $h_1'(s) = d\dfrac{e^{-as} - 1}{a}$, and taking $\varphi(s) = \frac{e^{-as}-1}{a}$, it is easy to check that $\varphi(0) = 0$ and $\varphi'(s) = -e^{-as} < 0$, for $s \in (0,1)$, thus $\varphi(s) < 0$, for $s \in (0,1)$, and the conclusion follows. If $d \le 0$, $h_1(0) = 1 > 0$ provides $h_1 > 0$ on $(0,1)$.

For $h_2(s) = \dfrac{1}{a}\left(1 - e^{-as} - cs + \dfrac{c}{a}(1 - e^{-as})\right)$, we obtain:

$$h_2'(s) = \frac{ae^{-as} - c + ce^{-as}}{a} =: \psi(s),$$

hence the sign of h_2' is the sign of ψ.

If $c = 0$, $h_2'(s) = e^{-as} > 0$, and h_2 is increasing on $(0,1)$. Now, consider $c \neq 0$, and calculate the derivative of ψ: $\psi'(s) = -ae^{-as} - ce^{-as} = -(a+c)e^{-as}$.

- If $a + c \leq 0$, then $\psi'(s) \geq 0$, and, using that $\psi(0) = 1 > 0$, we deduce that $\psi > 0$ and $h_2' > 0$ on $(0,1)$.
- If $a + c > 0$, then $\psi'(s) < 0$, so that $\psi = h_2'$ is decreasing on $(0,1)$. Since $\psi(0) = h_2'(0) = 1 > 0$ and $h_2'(1) = \psi(1) = \frac{ae^{-a}-c+ce^{-a}}{a} = \frac{(a+c)e^{-a}-c}{a}$, we get $h_2' \geq 0$ on $[0,1]$ if $\frac{(a+c)e^{-a}-c}{a} \geq 0$.

  Suppose that:

$$\frac{(a+c)e^{-a} - c}{a} < 0, \tag{A9}$$

  then if $c \leq 0$,

$$a > -c \geq 0, \text{ and } \frac{(a+c)e^{-a} - c}{a} > 0,$$

  which is contradiction. Hence $c > 0$ and h_2' has a change of sign at the point s_0 such that $(a+c)e^{-as_0} = c$, that is, $s_0 = -\frac{1}{a}\ln\frac{c}{a+c}$. In these hypotheses, $s_0 \in (0,1)$. Indeed, if $a > 0$, condition (A9) implies $(a+c)e^{-a} - c < 0$, thus $-a < \ln\left(\frac{c}{a+c}\right)$, and $s_0 < 1$. Besides, $0 < \frac{c}{a+c} < \frac{c}{c} = 1$, and $\ln\left(\frac{c}{a+c}\right) < 0$, which implies that $s_0 \in (0,1)$. If $a < 0$, condition (A9) implies $(a+c)e^{-a} > c$. Thus $-a > \ln\left(\frac{c}{a+c}\right)$ and $s_0 < 1$. On the other hand, $0 < a + c < c$, which implies $1 < \frac{c}{a+c}$, and $\ln\left(\frac{c}{a+c}\right) > 0$, obtaining $s_0 \in (0,1)$.

  This joint to $h_2'(0) = 1 > 0$ and $h_2'(1) < 0$ proves that h_2 is increasing on $(0, s_0)$ and decreasing on $(s_0, 1)$, and h_2 takes values between:

$$\min\left\{0, \frac{1}{a}\left(1 - e^{-a} - c + \frac{c}{a}(1 - e^{-a})\right)\right\}$$

  and

$$\begin{aligned} h_2(s_0) &= \frac{1}{a}\left(1 - \frac{c}{a+c} + \frac{c}{a}\ln\left(\frac{c}{a+c}\right) + \frac{c}{a}\left(1 - \frac{c}{a+c}\right)\right) \\ &= \frac{1}{a}\left(\frac{a}{a+c} + \frac{c}{a}\ln\left(\frac{c}{a+c}\right) + \frac{c}{a+c}\right) = \frac{1}{a}\left(1 + \frac{c}{a}\ln\frac{c}{a+c}\right). \end{aligned}$$

  Note that, even in this case, $h_2 \geq 0$ on $[0,1]$ if $h_2(1) \geq 0$.

□

Lemma A3. *If $a = b = 0$, the following properties hold:*

- *If $d < 0$, then $h_1' > 0$ on $(0,1)$.*
- *If $d > 0$, then $h_1' < 0$ on $(0,1)$.*
- *If $d = 0$, then $h_1' = 0$ on $(0,1)$.*
- *If $d \leq 0$, then $h_1 > 0$ on $(0,1)$.*
- *If $d > 0$, $h_1(1) = 1 - \frac{d}{2} \geq 0$, then $h_1 > 0$ on $(0,1)$.*
- *If $c \leq 1$, then $h_2' > 0$ on $(0,1)$, and $h_2 \geq 0$ on $[0,1]$.*
- *If $c > 1$, then $h_2' > 0$ in $(0, \frac{1}{c})$ and $h_2' < 0$ on $(\frac{1}{c}, 1)$. In this case, $h_2 \geq 0$ on $[0,1]$ if $c \leq 2$.*

Proof. For $h_1(s) = 1 - \dfrac{d}{2}s^2$, the conclusion is derived easily from $h_1'(s) = -ds$. Now, $h_2(s) = s - \dfrac{c}{2}s^2$, thus $h_2'(s) = 1 - cs$.

If $c \leq 1$, then $h_2'(s) = 1 - cs \geq 1 - s > 0$, for $s \in (0,1)$ and $h_2 > 0$ on $(0,1)$. However, for $c > 1$, $h_2' > 0$ on $(0, \frac{1}{c})$, and $h_2' < 0$ on $(\frac{1}{c}, 1)$, where $\frac{1}{c} \in (0,1)$. In this case, $h_2 \geq 0$ on $(0,1)$, if $\frac{c}{2} \leq 1$. □

Lemma A4. *If $b \neq 0$, $a^2 = 4b$, the following properties hold:*

- *If* $b+d<0$, *then* $h_1'>0$ *on* $(0,1)$.
- *If* $b+d>0$, *then* $h_1'<0$ *on* $(0,1)$. *In this case, if* $h_1(1)\geq 0$, *then* $h_1>0$ *on* $(0,1)$.
- *If* $b+d=0$, *then* $h_1'=0$ *on* $(0,1)$.
- *If* $b+d\leq 0$, *then* $h_1>0$ *on* $(0,1)$.
- *If* $\frac{a}{2}+c\leq 1$, *then* $h_2'>0$ *on* $(0,1)$, *and* $h_2\geq 0$ *on* $[0,1]$.
- *If* $\frac{a}{2}+c>1$, *then* $h_2'>0$ *in* $\left(0,\frac{1}{\frac{a}{2}+c}\right)$, *and* $h_2'<0$ *in* $\left(\frac{1}{\frac{a}{2}+c},1\right)$. *In this case, if* $h_2(1)\geq 0$, *then* $h_2>0$ *on* $(0,1)$.

Proof. For $h_1(s)=\left(1+\frac{d}{b}\right)\left(1+\frac{a}{2}s\right)e^{-\frac{a}{2}s}-\frac{d}{b}$, we get:

$$\begin{aligned} h_1'(s) &= \left(1+\frac{d}{b}\right)\left[\frac{a}{2}e^{-\frac{a}{2}s}+\left(1+\frac{a}{2}s\right)\left(-\frac{a}{2}\right)e^{-\frac{a}{2}s}\right] \\ &= e^{-\frac{a}{2}s}\left(1+\frac{d}{b}\right)\left(-\frac{a^2}{4}\right)s = e^{-\frac{a}{2}s}\left(1+\frac{d}{b}\right)(-b)s = -(b+d)se^{-\frac{a}{2}s}, \end{aligned}$$

and the conclusion follows.

For $h_2(s)=e^{-\frac{a}{2}s}\left[\frac{c}{b}\left(1+\frac{a}{2}s\right)+s\right]-\frac{c}{b}$, then,

$$\begin{aligned} h_2'(s) &= e^{-\frac{a}{2}s}\left[\frac{-ac}{2b}\left(1+\frac{a}{2}s\right)-\frac{a}{2}s+\frac{ac}{2b}+1\right] \\ &= e^{-\frac{a}{2}s}\left[1-\frac{a}{2}s-\frac{a^2c}{4b}s\right] = e^{-\frac{a}{2}s}\left[1-\frac{a}{2}s-cs\right] = e^{-\frac{a}{2}s}\left[1-\left(\frac{a}{2}+c\right)s\right]. \end{aligned}$$

If $\frac{a}{2}+c\leq 0$, then clearly $h_2'>0$. If $\frac{a}{2}+c>0$, then function $1-\left(\frac{a}{2}+c\right)s$ is equal to zero at $\bar{s}=\frac{1}{\frac{a}{2}+c}$ and $h_2'(s)>0$ for $s<\bar{s}$ and $h_2'(s)<0$ for $s>\bar{s}$. If $\frac{1}{\frac{a}{2}+c}\geq 1$ ($0<\frac{a}{2}+c\leq 1$), then $h_2'>0$ on $(0,1)$. On the other hand, if $\frac{a}{2}+c>1$, then $\bar{s}\in(0,1)$ and h_2 is nondecreasing in $\left(0,\frac{1}{\frac{a}{2}+c}\right)$ and nonincreasing in $\left(\frac{1}{\frac{a}{2}+c},1\right)$, taking values between $\min\{0,C_2=h_2(1)\}$ and,

$$h_2\left(\frac{1}{\frac{a}{2}+c}\right)=e^{-\frac{a}{a+2c}}\left(\frac{4c}{a^2}(a+c)+1\right)\frac{2}{a+2c}-\frac{4c}{a^2}.$$

□

Lemma A5. *If* $b\neq 0$, $a^2>4b$, *the following properties hold:*

- *If* $b+d<0$, *then* $h_1'>0$ *on* $(0,1)$.
- *If* $b+d>0$, *then* $h_1'<0$ *on* $(0,1)$. *In this case, if* $h_1(1)\geq 0$, *then* $h_1>0$ *on* $(0,1)$.
- *If* $b+d=0$, *then* $h_1'=0$ *on* $(0,1)$.
- *If* $b+d\leq 0$, *then* $h_1>0$ *on* $(0,1)$.
- *If* $a+2c\leq\sqrt{a^2-4b}\left(1+\frac{2}{e^{\sqrt{a^2-4b}}-1}\right)$, *then* $h_2'>0$ *on* $(0,1)$, *and* $h_2\geq 0$ *on* $[0,1]$.
- *If* $a+2c>\sqrt{a^2-4b}\left(1+\frac{2}{e^{\sqrt{a^2-4b}}-1}\right)$, *then* $h_2'>0$ *in* $(0,\widehat{y})$ *and* $h_2'<0$ *on* $(\widehat{y},1)$, *where*

$$\widehat{y}=\frac{1}{\alpha-\beta}\ln\left(\frac{c-\beta}{c-\alpha}\right)=\ln\left[\left(1+\frac{2\sqrt{a^2-4b}}{2c+a-\sqrt{a^2-4b}}\right)^{\frac{1}{\sqrt{a^2-4b}}}\right].$$

In these conditions, if $h_2(1)\geq 0$, *then* $h_2>0$ *on* $(0,1)$.

Proof. For $h_1(s) = \left(1+\frac{d}{b}\right)\frac{\beta e^{\alpha s} - \alpha e^{\beta s}}{\beta-\alpha} - \frac{d}{b}$, we get:

$$h_1'(s) = \frac{1}{\beta-\alpha}\left(1+\frac{d}{b}\right)\alpha\beta(e^{\alpha s} - e^{\beta s}), \; \forall s.$$

Using that:

$$\alpha\beta = \left(-\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}\right)\left(-\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}\right) = b \neq 0,$$

we obtain that:

$$h_1'(s) = \frac{1}{\beta-\alpha}(b+d)\,(e^{\alpha s} - e^{\beta s}), \; \forall s.$$

The conclusion is obtained taking into account that $\alpha > \beta$, thus $e^{\alpha s} - e^{\beta s} > 0$, for every s, and $\beta - \alpha < 0$.

For, $h_2(s) = \dfrac{\left(\frac{\beta c}{b} - 1\right)e^{\alpha s} + \left(1 - \frac{\alpha c}{b}\right)e^{\beta s}}{\beta - \alpha} - \dfrac{c}{b}$, we get:

$$\begin{aligned} h_2'(s) &= \frac{1}{\beta-\alpha}\left[\left(\frac{\alpha\beta c}{b} - \alpha\right)e^{\alpha s} + \left(\beta - \frac{\alpha\beta c}{b}\right)e^{\beta s}\right] \\ &= \frac{1}{\beta-\alpha}\left[(c-\alpha)\,e^{\alpha s} + (\beta - c)\,e^{\beta s}\right] = \frac{1}{\beta-\alpha}e^{\beta s}\left[(c-\alpha)\,e^{(\alpha-\beta)s} + (\beta - c)\right]. \end{aligned}$$

Take $\rho(s) = (c-\alpha)\,e^{(\alpha-\beta)s} + (\beta - c)$, which satisfies $\rho(0) = \beta - \alpha < 0$, $\rho(1) = (c-\alpha)\,e^{(\alpha-\beta)} + (\beta - c)$, and

$$\rho'(s) = (c-\alpha)\,(\alpha-\beta)e^{(\alpha-\beta)s}, \; \forall s.$$

If $c \le \alpha$, then $\rho' \le 0$ and $\rho < 0$ on $(0,1)$. If $c > \alpha$, then $\rho' > 0$ on $(0,1)$ but condition $\rho(1) \le 0$, that is,

$$(c-\alpha)\,e^{(\alpha-\beta)} + (\beta - c) \le 0 \tag{A10}$$

implies $\rho < 0$ on $(0,1)$. In these two cases, $h_2' > 0$ on $(0,1)$. Condition $c \le \alpha$ is equivalent to $c + \frac{a}{2} \le \sqrt{\left(\frac{a}{2}\right)^2 - b}$. On the other hand, consider $c > \alpha$, that is, $c + \frac{a}{2} > \sqrt{\left(\frac{a}{2}\right)^2 - b}$. In this case, estimate (A10) is equivalent to:

$$e^{(\alpha-\beta)} \le \frac{c-\beta}{c-\alpha} > 0.$$

Hence,

$$(\alpha - \beta) \le \ln\left(\frac{c-\beta}{c-\alpha}\right),$$

and

$$1 \le \frac{1}{\alpha-\beta}\ln\left(\frac{c-\beta}{c-\alpha}\right).$$

Another expression equivalent to (A10) is:

$$e^{\sqrt{a^2-4b}} \le \frac{2c + a + \sqrt{a^2-4b}}{2c + a - \sqrt{a^2-4b}} = 1 + \frac{2\sqrt{a^2-4b}}{2c + a - \sqrt{a^2-4b}},$$

or, using that $c > \alpha$, that is, $2c + a - \sqrt{a^2-4b} > 0$,

$$2c + a - \sqrt{a^2-4b} \le \frac{2\sqrt{a^2-4b}}{e^{\sqrt{a^2-4b}} - 1},$$

and

$$2c + a \le \sqrt{a^2 - 4b}\left(1 + \frac{2}{e^{\sqrt{a^2-4b}} - 1}\right).$$

Function ρ is equal to zero exactly at the point $\widehat{y} = \frac{1}{\alpha-\beta}\ln\left(\frac{c-\beta}{c-\alpha}\right)$. If $c > \alpha$ and $(c-\alpha)\,e^{(\alpha-\beta)} + (\beta - c) > 0$, then $\widehat{y} \in (0,1)$, thus $h_2' > 0$ in $(0, \widehat{y})$ and $h_2' < 0$ on $(\widehat{y}, 1)$. Here:

$$\begin{aligned}
\widehat{y} = \frac{1}{\alpha-\beta}\ln\left(\frac{c-\beta}{c-\alpha}\right) &= \ln\left[\left(\frac{c-\beta}{c-\alpha}\right)^{\frac{1}{\alpha-\beta}}\right] \\
&= \ln\left[\left(\frac{2c + a + \sqrt{a^2-4b}}{2c + a - \sqrt{a^2-4b}}\right)^{\frac{1}{\sqrt{a^2-4b}}}\right] \\
&= \ln\left[\left(1 + \frac{2\sqrt{a^2-4b}}{2c + a - \sqrt{a^2-4b}}\right)^{\frac{1}{\sqrt{a^2-4b}}}\right].
\end{aligned}$$

The condition which guarantees that $\widehat{y} \in (0,1)$ can be written as:

$$a + 2c > \sqrt{a^2 - 4b}\left(1 + \frac{2}{e^{\sqrt{a^2-4b}} - 1}\right).$$

□

Lemma A6. *If $b \neq 0$, $a^2 < 4b$, the following properties hold:*

- *If $b + d < 0$ and $\sqrt{b - \frac{a^2}{4}} \le \pi$, then $h_1' > 0$ on $(0,1)$.*
- *If $b + d > 0$ and $\sqrt{b - \frac{a^2}{4}} \le \pi$, then $h_1' < 0$ on $(0,1)$. In this case, if*

$$(b+d)\,e^{-\frac{a}{2}}\left\{\cos\sqrt{b - \frac{a^2}{4}} + \frac{a}{2\sqrt{b - \frac{a^2}{4}}}\sin\sqrt{b - \frac{a^2}{4}}\right\} \ge d,$$

  then $h_1 > 0$ on $(0,1)$.
- *If $b + d = 0$, then $h_1' = 0$ on $(0,1)$.*
- *If $b + d = 0$; or [$b + d < 0$ and $\sqrt{b - \frac{a^2}{4}} \le \pi$], then $h_1 > 0$ on $(0,1)$.*
- *Suppose that one of the following conditions holds:*

  * $\sqrt{b - \frac{a^2}{4}} \le \frac{\pi}{2}$ *and* $\frac{a}{2} + c \le 0$*; or*
  * $\sqrt{b - \frac{a^2}{4}} < \frac{\pi}{2}$ *and* $0 < \frac{a}{2} + c \le \sqrt{b - \frac{a^2}{4}}\cot\left(\sqrt{b - \frac{a^2}{4}}\right)$*; or*
  * $\frac{\pi}{2} < \sqrt{b - \frac{a^2}{4}} < \pi$*, and* $\frac{a}{2} + c \le \sqrt{b - \frac{a^2}{4}}\cot\left(\sqrt{b - \frac{a^2}{4}}\right) < 0$*,*

  then $h_2' > 0$ on $(0,1)$, and $h_2 \ge 0$ on $[0,1]$.
- *If one of the following conditions holds:*

  * $\sqrt{b - \frac{a^2}{4}} = \frac{\pi}{2}$ *and* $\frac{a}{2} + c > 0$*; or*
  * $\sqrt{b - \frac{a^2}{4}} < \frac{\pi}{2}$ *and* $\frac{a}{2} + c > \sqrt{b - \frac{a^2}{4}}\cot\left(\sqrt{b - \frac{a^2}{4}}\right)$*; or*
  * $\sqrt{b - \frac{a^2}{4}} \in (\frac{\pi}{2}, \pi]$*, and* $\frac{a}{2} + c > 0$*,*

* $\sqrt{b-\frac{a^2}{4}} \in (\pi, \frac{3\pi}{2})$, *and* $0 < \frac{a}{2}+c \leq \sqrt{b-\frac{a^2}{4}}\cot\left(\sqrt{b-\frac{a^2}{4}}\right)$,

then $h_2' > 0$ *on* $(0, s_0)$, *and* $h_2' < 0$ *on* $(s_0, 1)$, *where:*

$$s_0 = \frac{1}{\sqrt{b-\frac{a^2}{4}}}\arctan\left(\frac{\sqrt{b-\frac{a^2}{4}}}{\frac{a}{2}+c}\right).$$

Under these hypotheses, if $h_2(1) \geq 0$, *then* $h_2 > 0$ *on* $(0, 1)$.

- *If one of the following conditions hold:*

* $\frac{3\pi}{2} \geq \sqrt{b-\frac{a^2}{4}} \geq \pi$, *and* $\frac{a}{2}+c < 0$; *or*
* $\frac{\pi}{2} < \sqrt{b-\frac{a^2}{4}} < \pi$, *and* $0 > \frac{a}{2}+c > \sqrt{b-\frac{a^2}{4}}\cot\left(\sqrt{b-\frac{a^2}{4}}\right)$,

then $h_2' > 0$ *on* $[0, s_1)$ *and* $h_2' < 0$ *on* $(s_1, 1)$, *where:*

$$s_1 = \frac{1}{\sqrt{b-\frac{a^2}{4}}}\left(\arctan\left(\frac{\sqrt{b-\frac{a^2}{4}}}{\frac{a}{2}+c}\right)+\pi\right).$$

Under these hypotheses, if $h_2(1) \geq 0$, *then* $h_2 > 0$ *on* $(0, 1)$.

- *If one of the following conditions hold:*

* $\sqrt{b-\frac{a^2}{4}} \in (\pi, \frac{3\pi}{2})$, *and* $\frac{a}{2}+c > \sqrt{b-\frac{a^2}{4}}\cot\left(\sqrt{b-\frac{a^2}{4}}\right) > 0$; *or*
* $\sqrt{b-\frac{a^2}{4}} = \frac{3\pi}{2}$, *and* $\frac{a}{2}+c > 0$,

then $h_2' > 0$ *on* $(0, s_0)$, $h_2' < 0$ *on* (s_0, s_1), *and* $h_2' > 0$ *on* $(s_1, 1)$. *Moreover, if* $h_2(s_1) \geq 0$, *then* $h_2 \geq 0$ *on* $(0, 1)$, *and if* $h_2(s_1) > 0$, *then* $h_2 > 0$ *on* $(0, 1)$.

- *If* $\frac{a}{2}+c = 0$, *then* $h_2' > 0$ *in the interval* $\left((2k-1)\frac{\pi}{\sqrt{4b-a^2}}, (2k+1)\frac{\pi}{\sqrt{4b-a^2}}\right)$, *for* $k \in \mathbb{Z}$ *even, and* $h_2' < 0$ *in that interval for* $k \in \mathbb{Z}$ *odd. In consequence, if* $\frac{a}{2}+c = 0$, *and* $\sqrt{b-\frac{a^2}{4}} \leq \frac{\pi}{2}$, *then* $h_2' > 0$ *on* $(0, 1)$.

Proof. Let $R := \sqrt{b-\frac{a^2}{4}}$. By Lemma A1, since:

$$h_1(s) = \left(1+\frac{d}{b}\right)e^{-\frac{a}{2}s}\left\{\cos Rs + \frac{a}{2R}\sin Rs\right\} - \frac{d}{b},$$

and $4R^2 + a^2 = 4b$, we have:

$$\begin{aligned} h_1'(s) &= \left(1+\frac{d}{b}\right)\left[-\frac{a}{2}e^{-\frac{a}{2}s}\left\{\cos Rs + \frac{a}{2R}\sin Rs\right\}\right. \\ &\qquad \left. + e^{-\frac{a}{2}s}\left\{-R\sin Rs + \frac{a}{2}\cos Rs\right\}\right] \\ &= \left(1+\frac{d}{b}\right)e^{-\frac{a}{2}s}(-1)\left[\frac{a^2}{4R}+R\right]\sin Rs \\ &= -\frac{b+d}{R}e^{-\frac{a}{2}s}\sin Rs. \end{aligned}$$

Note that, in this case, $0 \le a^2 < 4b$, thus $b > 0$. The study of the sign of h_1' is reduced to the study of the sign of $b + d$ and the function $s \to \sin Rs$. The assertions concerning h_1 are, therefore, valid. Notice that we consider $R = \sqrt{b - \frac{a^2}{4}} \le \pi$ in order to keep the sign of $\sin \sqrt{b - \frac{a^2}{4}} s$ positive on $(0, 1)$.

Now, we consider:

$$h_2(s) = e^{-\frac{a}{2}s} \left\{ \frac{c}{b} \cos \sqrt{b - \frac{a^2}{4}} s + \frac{1 + \frac{ac}{2b}}{\sqrt{b - \frac{a^2}{4}}} \sin \sqrt{b - \frac{a^2}{4}} s \right\} - \frac{c}{b},$$

and denote again by $R := \sqrt{b - \frac{a^2}{4}}$, thus, the derivative of h_2 is the function:

$$\begin{aligned} h_2'(s) &= e^{-\frac{a}{2}s} \left[-\frac{ac}{2b} \cos Rs - \frac{a}{2} \frac{\left(1 + \frac{ac}{2b}\right)}{R} \sin Rs \right. \\ &\qquad \left. -\frac{c}{b} R \sin Rs + \left(1 + \frac{ac}{2b}\right) \cos Rs \right] \\ &= e^{-\frac{a}{2}s} \left[\cos Rs + \left(-\frac{a}{2R} \left(1 + \frac{ac}{2b}\right) - \frac{cR}{b} \right) \sin Rs \right] =: e^{-\frac{a}{2}s} \varphi(s), \ \forall s. \end{aligned}$$

To study the sign of h_2', we analyze the sign of φ.

If $a + 2c = 0$, then:

$$-\frac{a}{2R}\left(1 + \frac{ac}{2b}\right) - \frac{cR}{b} = \frac{c}{R}\left(1 - \frac{c^2}{b}\right) - \frac{cR}{b} = \frac{c}{b}\left(\frac{b - c^2}{R} - R\right) = \frac{c}{b}\left(\frac{\frac{a^2}{4} - c^2}{R}\right) = 0,$$

and $\varphi(s) = \cos Rs$, for every s, and assuming that $R \le \frac{\pi}{2}$, we get that $\varphi(s) > 0$ on $(0, 1)$, thus $h_2' > 0$ on $(0, 1)$.

Now, suppose that:

$$N := \frac{a}{2R}\left(1 + \frac{ac}{2b}\right) + \frac{cR}{b} \ne 0,$$

(which is equivalent to $\frac{a}{2} + c \ne 0$, since $RN = \frac{a}{2}(1 + \frac{ac}{2b}) + \frac{cR^2}{b} = \frac{a}{2} + c$), and calculate the zeros of $\varphi(s) := \cos Rs - N \sin Rs$. Equation $\varphi(s) = 0$ is equivalent to:

$$\cos Rs = N \sin Rs. \tag{A11}$$

For a solution $s \in \mathbb{R}$ to this equation, it is not possible that $\cos(Rs) = 0$. Indeed, if $\cos(Rs) = 0$, then $N \sin Rs = 0$, and thus $\sin Rs = 0$, which is absurd. Hence, Equation (A11) is equivalent to

$$\tan Rs = \frac{1}{N}.$$

Then, there exists an infinite number of solutions:

$$s_k = \frac{1}{R} \arctan\left(\frac{1}{N}\right) \in \left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right), \ k \in \mathbb{Z}.$$

If we denote by $\arctan\left(\frac{1}{N}\right)$ the unique number in $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $\tan\left(\arctan\left(\frac{1}{N}\right)\right) = \frac{1}{N}$, then,

$$s_k = \frac{1}{R}\left(\arctan\left(\frac{1}{N}\right) + k\pi\right), \ k \in \mathbb{Z}.$$

Notice that $s_0 = \frac{1}{R} \arctan\left(\frac{1}{N}\right) \in (-\frac{\pi}{2R}, \frac{\pi}{2R})$ and $s_0 \ne 0$. Besides, if $N > 0$, then $s_0 > 0$, and if $N < 0$, $s_0 < 0$.

Suppose that $R \leq \frac{\pi}{2}$. In this case, $-\frac{\pi}{2R} \leq -1 < 0 < 1 \leq \frac{\pi}{2R}$, and the unique point in $(0,1)$ where φ may change its sign is at most $s_0 \neq 0$. Since $\varphi(0) = 1 > 0$, if $s_0 < 0$ or $s_0 \geq 1$, then $\varphi > 0$ on $(0,1)$ and $h_2' > 0$ on $(0,1)$.

Next, we find sufficient conditions for the validity of $s_0 < 0$, or $s_0 \geq 1$. Note that $s_0 < 0$ if and only if $N < 0$, and this is equivalent to $\frac{a}{2} + c < 0$. This follows from $N = \frac{1}{R}(\frac{a}{2} + c)$, and $R > 0$. This proves that $R \leq \frac{\pi}{2}$ and $\frac{a}{2} + c < 0$ imply that $h_2' > 0$ on $(0,1)$.

On the other hand, $s_0 \geq 1$ is possible for $R < \frac{\pi}{2}$, and it is equivalent to $\arctan\left(\frac{1}{N}\right) \geq R > 0$. If $R < \frac{\pi}{2}$, the nondecreasing character of function tan in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ produces $\frac{1}{N} \geq \tan(R) > 0$. Besides, $\frac{1}{N} > 0$ and $0 < N \leq \cot(R)$, thus $0 < \dfrac{a}{2} + c \leq R\cot(R)$, and the conclusion follows.

Consider the case where $s_0 \in (0,1)$, that is, $\arctan\left(\frac{1}{N}\right) \in (0,R)$, or equivalently, $N > 0$ and $\arctan\left(\frac{1}{N}\right) < R$. These properties hold if $R = \frac{\pi}{2}$ and $\frac{a}{2} + c > 0$. Indeed, $N = \frac{1}{R}\left(\frac{a}{2} + c\right) > 0$ and the second condition is trivially valid. On the other hand, if $0 < R < \frac{\pi}{2}$, and $\frac{a}{2} + c > R\cot(R)$, then $N = \frac{1}{R}\left(\frac{a}{2} + c\right) > 0$ and $N > \cot(R)$, that is, $0 < \frac{1}{N} < \tan(R)$ and $0 < \arctan\left(\frac{1}{N}\right) < R$. Under these assumptions, and, using that $\varphi(0) = 1 > 0$ and

$$\varphi\left(\frac{\pi}{2R}\right) = \cos R\frac{\pi}{2R} - N\sin R\frac{\pi}{2R} = -N < 0,$$

we deduce that φ has a change of sign in $(0,1)$, exactly at the point s_0, obtaining $\varphi > 0$ on $(0,s_0)$, and $\varphi < 0$ on $(s_0,1)$. This proves that $h_2' > 0$ on $(0,s_0)$ and $h_2' < 0$ on $(s_0,1)$.

Now, we consider the case $R > \frac{\pi}{2}$, then $\frac{\pi}{2R} < 1$, and there could be two zeros of φ in $(0,1)$, but at most one zero in $(0, \frac{\pi}{2R})$. Assuming that $s_0 < 0$ (that is $N < 0$, or equivalently $\frac{a}{2} + c < 0$), we obtain that $\varphi > 0$ on $[0, \frac{\pi}{2R}]$. If $s_1 \geq 1$, there are not changes of sign for φ in $(0,1)$, hence $\varphi > 0$ on $(0,1)$ and $h_2' > 0$ on $(0,1)$. This occurs if:

$$s_1 = \frac{1}{R}\left(\arctan\left(\frac{1}{N}\right) + \pi\right) \geq 1,$$

which is equivalent to:

$$\arctan\left(\frac{1}{N}\right) \geq R - \pi > -\frac{\pi}{2}.$$

Under these assumptions, $N < 0$, then $0 > \arctan\left(\frac{1}{N}\right) \geq R - \pi > -\frac{\pi}{2}$, which is a contradiction if $R \geq \pi$ and, for $\frac{\pi}{2} < R < \pi$, this inequality is reduced, by the non-decreasing character of tan, to:

$$0 > \frac{1}{N} \geq \tan(R - \pi) = \tan(R).$$

This expression means $N \leq \cot(R) < 0$, and $\frac{a}{2} + c \leq R\cot(R) < 0$. That is, for $R \in \left(\frac{\pi}{2}, \pi\right)$ and $\frac{a}{2} + c \leq R\cot(R) < 0$, then $h_2' > 0$ on $(0,1)$.

If $R \geq \pi$ and $\frac{a}{2} + c < 0$, then it is clear that $\frac{\pi}{2R} < s_1 < 1$. If $\frac{\pi}{2} < R < \pi$ and $0 > \frac{a}{2} + c > R\cot(R)$, following a similar procedure, we get $0 < s_1 < 1$.

Taking into account that, in this case,

$$\varphi\left(\frac{3\pi}{2R}\right) = \cos R\frac{3\pi}{2R} - N\sin R\frac{3\pi}{2R} = N < 0,$$

then $\varphi > 0$ on $[0,s_1)$ and $\varphi < 0$ on $(s_1,1)$, hence the same behavior follows for h_2', thus h_2' has a change of sign in $(0,1)$.

If $R > \frac{\pi}{2}$, and $\frac{a}{2}+c > 0$, then $N > 0$ and $s_0 > 0$. Note that $\varphi(0) = 1 > 0$ and $\varphi(\frac{\pi}{2R}) = \cos\frac{\pi}{2} - N\sin\frac{\pi}{2} = -N < 0$. If $R \in (\frac{\pi}{2}, \pi]$ and $\frac{a}{2}+c > 0$, then $R-\pi \le 0 < \arctan\left(\frac{1}{N}\right)$, thus $\arctan\left(\frac{1}{N}\right) + \pi \ge R$, and $s_1 \ge 1$. This proves that $\varphi > 0$ on $(0, s_0)$, and $\varphi < 0$ on $(s_0, 1)$.

On the other hand, if $R \in (\pi, \frac{3\pi}{2})$, and $\frac{a}{2}+c > R\cot(R) > 0$, then $\cot(R) < N$, and $1 < N\tan(R)$, which implies that $\tan(R) = \tan(R-\pi) > \frac{1}{N} > 0$, and, using that $R - \pi < \frac{\pi}{2}$, we get $0 < \arctan\left(\frac{1}{N}\right) < R-\pi$, which implies that $\arctan\left(\frac{1}{N}\right)+\pi < R$, in consequence $s_1 = \frac{1}{R}\left[\arctan\left(\frac{1}{N}\right)+\pi\right] < 1$. Note that $\varphi(\frac{3\pi}{2R}) = N > 0$. In this case, we deduce that $\varphi > 0$ on $(0, s_0)$, $\varphi < 0$ on (s_0, s_1), and $\varphi > 0$ on $(s_1, 1)$, and analogously for h_2'. If $h_2(s_1) \ge 0$, then it is clear that $h_2 \ge 0$ on $(0,1)$. Note that, if $R = \frac{3\pi}{2}$, then obviously $s_1 < 1$. Finally, if $R \in (\pi, \frac{3\pi}{2})$, and $0 < \frac{a}{2}+c \le R\cot(R)$, then it is satisfied $s_1 \ge 1$, therefore $\varphi > 0$ on $(0, s_0)$, and $\varphi < 0$ on $(s_0, 1)$. $\square$

For $\sqrt{b-\frac{a^2}{4}} \le \frac{3\pi}{2}$, only s_0 and s_1 may belong to $(0,1)$. The value of $R := \sqrt{b-\frac{a^2}{4}}$ defined in the proof of Lemma A6 can be arbitrarily large, in such a way that there could be many zeros of h_2' in $(0,1)$. In this sense, the following Remarks are of interest, where it is denoted:

$$N := \frac{a}{2R}\left(1+\frac{ac}{2b}\right)+\frac{cR}{b}, \text{ and } \varphi(s) := \cos Rs - N\sin Rs.$$

Remark A1. *Consider the case $b \neq 0$, $a^2 < 4b$. If $\frac{a}{2}+c = 0$, then $N = 0$, and, therefore,*

$$\varphi(s) = \cos Rs - N\sin Rs = \cos Rs.$$

In consequence, it is trivially satisfied that $\varphi > 0$ if $Rs \in \left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$, where $k \in \{0\} \cup \{2n \mid n \in \mathbb{N}\} \cup \{-2n \mid n \in \mathbb{N}\}$, that is, if

$$s \in \left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right), \qquad k \in \{0\} \cup \{2n \mid n \in \mathbb{N}\} \cup \{-2n \mid n \in \mathbb{N}\}.$$

This proves that $h_2' > 0$ in:

$$\left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right), \qquad k \in \{0\} \cup \{2n \mid n \in \mathbb{N}\} \cup \{-2n \mid n \in \mathbb{N}\},$$

and, similarly, $h_2' < 0$ in

$$\left((2k+1)\frac{\pi}{2R}, (2k+3)\frac{\pi}{2R}\right), \qquad k \in \{0\} \cup \{2n \mid n \in \mathbb{N}\} \cup \{-2n \mid n \in \mathbb{N}\},$$

or, equivalently, $h_2' < 0$ in: $\left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right)$, for k odd.

Therefore, if $h_2((2k+1)\frac{\pi}{2R}) \ge 0$, for every $k \in \mathbb{Z}$ odd with $(2k+1)\frac{\pi}{2R} \in (0,1)$, then $h_2 \ge 0$ on $(0,1)$.

Note that k odd with $(2k+1)\frac{\pi}{2R} \in (0,1)$, means k odd with $-\frac{1}{2} < k < \frac{R}{\pi} - \frac{1}{2}$, that is, $k = 1, 3, 5, \ldots$ with $k < \frac{R}{\pi} - \frac{1}{2}$. Obviously, the set of such elements k is nonempty if $R > \frac{3\pi}{2}$.

Remark A2. *Consider the case $b \neq 0$, $a^2 < 4b$, and suppose that $\frac{a}{2}+c \neq 0$, that is, $N \neq 0$, then*

$$\varphi(s) = \cos Rs - N\sin Rs = 0$$

if and only if:

$$s = s_k = \frac{1}{R}\left(\arctan\left(\frac{1}{N}\right)+k\pi\right) = s_0 + \frac{k\pi}{R} \in \left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right), \, k \in \mathbb{Z}.$$

Note that:

$$\varphi\left((2k-1)\frac{\pi}{2R}\right) = \cos(2k-1)\frac{\pi}{2} - N\sin(2k-1)\frac{\pi}{2} = \begin{cases} -N, & \textit{if } k \textit{ is odd,} \\ N, & \textit{if } k \textit{ is even.} \end{cases}$$

Function φ is obviously $\frac{2\pi}{R}$-periodic. We distinguish two cases:

- *If $N > 0$, that is, $\frac{a}{2} + c > 0$, then $\varphi > 0$ on (s_{2k-1}, s_{2k}), for $k \in \mathbb{Z}$, and $\varphi < 0$ on (s_{2k}, s_{2k+1}), for $k \in \mathbb{Z}$. In Figure A1, we show the graph of the function φ for $R = 1$ and $N = 2$.*

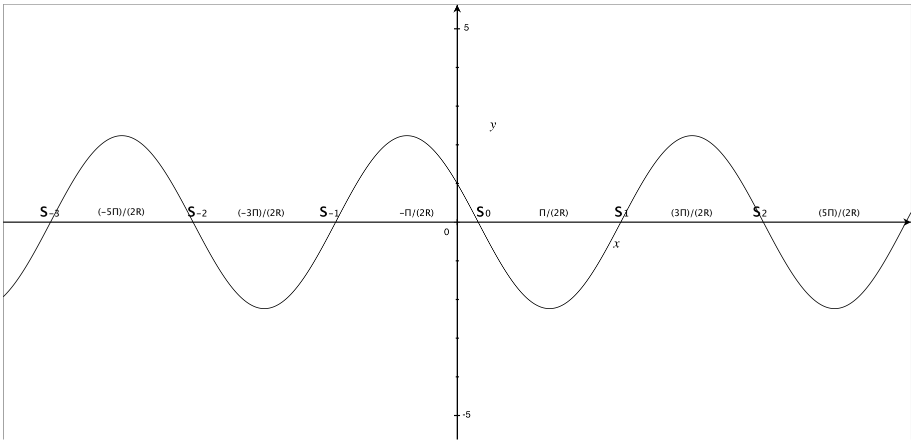

Figure A1. Graph of the function φ for $R = 1$ and $N = 2$.

- *If $N < 0$, that is, $\frac{a}{2} + c < 0$, then $\varphi > 0$ on (s_{2k}, s_{2k+1}), for $k \in \mathbb{Z}$, and $\varphi < 0$ on (s_{2k-1}, s_{2k}), for $k \in \mathbb{Z}$. In Figure A2, we show the graph of the function φ for $R = 1$ and $N = -2$.*

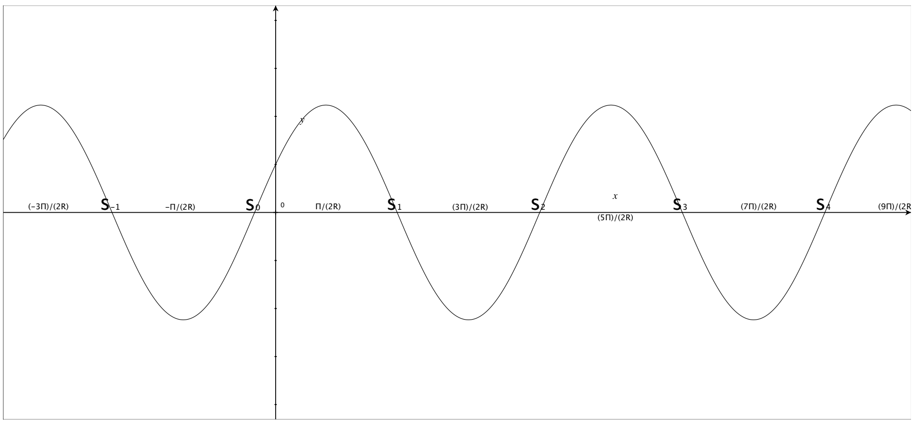

Figure A2. Graph of the function φ for $R = 1$ and $N = -2$.

Hence, to prove that $h_2 \geq 0$ on $(0,1)$, it is enough to check that:

$$h_2(s_k) \geq 0, \textit{ for every } k \textit{ odd with } s_k \in (0,1), \qquad \textit{if } N > 0, \tag{A12}$$

and

$$h_2(s_k) \geq 0, \textit{ for every } k \textit{ even with } s_k \in (0,1), \qquad \textit{if } N < 0. \tag{A13}$$

We calculate the values of $k \in \mathbb{Z}$ with $s_k \in (0,1)$. It could happen for $s_0 \in \left(-\frac{\pi}{2R}, \frac{\pi}{2R}\right)$, or s_k with $k \geq 1$.

We check that $s_0 > 0$ if and only if $N > 0$. Indeed, for $k = 0$, $s_0 = \frac{1}{R}\arctan\left(\frac{1}{N}\right) > 0$ if and only if $N > 0$, that is $\frac{a}{2} + c > 0$. We have already studied that $s_0 \in (0,1)$ in the cases:

1. $R \geq \frac{\pi}{2}$ and $\frac{a}{2} + c > 0$; or
2. $R < \frac{\pi}{2}$ and $\frac{a}{2} + c > R\cot(R) > 0$.

On the other hand, for $k \geq 1$, $s_k > \frac{(2k-1)\pi}{2R} \geq \frac{\pi}{2R} > 0$. *In consequence, for the validity of* $s_k < 1$ *is necessary that* $R > \frac{(2k-1)\pi}{2}$. *Besides, the condition* $s_k < 1$ *is satisfied if* $\arctan\left(\frac{1}{N}\right) + k\pi < R$, *that is,*

$$\arctan\left(\frac{1}{N}\right) < R - k\pi. \tag{A14}$$

We distinguish two cases:

Case 1: $N > 0$. *In this case,* $\arctan\left(\frac{1}{N}\right) > 0$, *and we are interested in the values of k odd. Hence, necessarily,* $R - k\pi > 0$ ($k < \frac{R}{\pi}$) *and we distinguish again two subcases:*

Subcase 1.1: *If* $R - k\pi \geq \frac{\pi}{2}$, (A14) *is trivially satisfied, and the values of k are those with* $R - \frac{\pi}{2} \geq k\pi$, *that is,* $\frac{R}{\pi} - \frac{1}{2} \geq k$ *or* $(2k+1)\frac{\pi}{2} \leq R$.

Subcase 1.2: *If* $0 < R - k\pi < \frac{\pi}{2}$, *which is equivalent to* $\frac{R}{\pi} - \frac{1}{2} < k < \frac{R}{\pi}$, *or* $k\pi < R < \frac{\pi}{2} + k\pi = (2k+1)\frac{\pi}{2}$, *then* (A14) *is equivalent to*

$$\frac{1}{N} < \tan(R - k\pi) = \tan(R),$$

or

$$\frac{1}{R}\left(\frac{a}{2} + c\right) = N > \cot(R),$$

which is written as

$$\frac{a}{2} + c > R\cot(R) > 0.$$

Summarizing, for $N > 0$ ($\frac{a}{2} + c > 0$), *the admissible values are* s_k *with* $1 \leq k \leq \frac{R}{\pi} - \frac{1}{2}$, *or in the case* $\frac{a}{2} + c > R\cot(R) > 0$, *the values of* $k \geq 1$ *such that* $\frac{R}{\pi} - \frac{1}{2} < k < \frac{R}{\pi}$, *that is,* $R \in (k\pi, (2k+1)\frac{\pi}{2})$. *As indicated, to study the sign of* h_2, *we are interested in the values of this function at the corresponding points* s_k *in* $(0,1)$ *with k odd.*

Case 2: $N < 0$ ($\frac{a}{2} + c < 0$). *In this case, the values of k interesting for the study of the sign of* h_2 *are even (and positive, since* $s_0 < 0$). *We check that the values of k with* $s_k \in (0,1)$ *are:*

1. $1 \leq k \leq \frac{R}{\pi}$; or
2. $1 \leq k$, $\frac{R}{\pi} < k < \frac{R}{\pi} + \frac{1}{2}$, that is, $R \in \left((2k-1)\frac{\pi}{2}, k\pi\right)$, provided that $R\cot(R) < \frac{a}{2} + c < 0$.

Indeed, $\arctan\left(\frac{1}{N}\right) < 0$, *and consider two subcases:*

Subcase 2.1: *If* $R \geq k\pi$, ($1 \leq k \leq \frac{R}{\pi}$), (A14) *is trivially satisfied.*

Subcase 2.2: *If* $R - k\pi < 0$, *then condition* (A14) *implies necessarily that* $R - k\pi > -\frac{\pi}{2}$. *Assuming that* $-\frac{\pi}{2} < R - k\pi < 0$, *that is,* $R \in \left((2k-1)\frac{\pi}{2}, k\pi\right)$, *which corresponds to the values of* $k \geq 1$ *with*

$$\frac{R}{\pi} < k < \frac{R}{\pi} + \frac{1}{2},$$

and (A14) *is valid since* $\frac{1}{N} < \tan(R - k\pi) = \tan(R) < 0$, *that is,* $\cot(R) < N < 0$, *is equivalent to* $R\cot(R) < \frac{a}{2} + c < 0$. *To study the sign of* h_2, *we calculate the values of this function at the points* $s_k \in (0,1)$ *with k even.*

Remark A3. *In the case* $b \neq 0$, $a^2 < 4b$, *note that* $h_2'(s) = e^{-\frac{a}{2}s}\varphi(s)$, *for every s, where the function* φ *given in the proof of Lemma A6 is* $\frac{2\pi}{R}$*-periodic. However,* h_2' *is not periodic if* $a \neq 0$. *In fact, for* $a > 0$, *the oscillations of* h_2' *are damped, and, for* $a < 0$, *the oscillations have increasing amplitude. For instance, in Figure A3, we show the graph of the function* $h_2'(x) = e^{-\frac{a}{2}x}\varphi(x)$, *for* $a = 3$, $R = 1$, *and* $N = 2$.

We obtain $h_2(s) = h_2(0) + \int_0^s e^{-\frac{a}{2}r}\varphi(r)\,dr = \int_0^s e^{-\frac{a}{2}r}\varphi(r)\,dr$. *In consequence,* $h_2 \geq 0$ *on* $(0,1)$ *if* $\int_0^s e^{-\frac{a}{2}r}\varphi(r)\,dr = \int_0^s e^{-\frac{a}{2}r}\{\cos Rr - N\sin Rr\}\,dr \geq 0$ *on* $(0,1)$. *See Figure A3 for the corresponding regions in the particular case where* $a = 3$, $R = 1$, *and* $N = 2$.

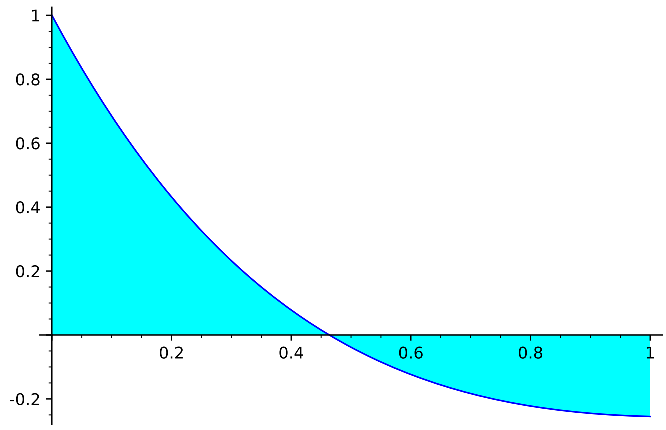

Figure A3. Graph of the function $h_2'(x) = e^{-\frac{a}{2}x}\varphi(x)$, for $a = 3$, $R = 1$, and $N = 2$, with the corresponding regions.

Equivalently, $\int_0^s e^{-\frac{a}{2}r}\cos Rr\,dr \ \geq N\int_0^s e^{-\frac{a}{2}r}\sin Rr\,dr$, *for* $s \in (0,1)$. *Integrating by parts, we get the equivalent expression:*

$$\int_0^s e^{-\frac{a}{2}r}\cos Rr\,dr \ = e^{-\frac{a}{2}s}\frac{\sin Rs}{R} + \frac{a}{2R}\int_0^s e^{-\frac{a}{2}r}\sin Rr\,dr \geq N\int_0^s e^{-\frac{a}{2}r}\sin Rr\,dr,$$

for $s \in (0,1)$, *or,*

$$e^{-\frac{a}{2}s}\sin Rs \geq c\int_0^s e^{-\frac{a}{2}r}\sin Rr\,dr,\ \forall s \in (0,1).$$

Taking $\mu(s) = e^{-\frac{a}{2}s}\sin Rs - c\int_0^s e^{-\frac{a}{2}r}\sin Rr\,dr$, *and proceeding similarly to Remark A2, conditions (A12) and (A13) can be written with this formulation in the following equivalent way:*

$$e^{-\frac{a}{2}s_k}\sin Rs_k \geq c\int_0^{s_k} e^{-\frac{a}{2}r}\sin Rr\,dr,\ \textit{for every } k \textit{ odd with } s_k \in (0,1),\ \textit{ if } N > 0,$$

and

$$e^{-\frac{a}{2}s_k}\sin Rs_k \geq c\int_0^{s_k} e^{-\frac{a}{2}r}\sin Rr\,dr,\ \textit{for every } k \textit{ even with } s_k \in (0,1),\ \textit{ if } N < 0.$$

Appendix C. Technical Results Concerning the Properties of Function g

Lemma A7. *If* $b = 0$, $a \neq 0$, *then* $g > 0$ *and* $g' > 0$ *on* $(0,1)$.

Proof. In this case, $g(z) = \frac{1-e^{-az}}{a}$, and it is easy to check that $g(0) = 0$, $g'(z) = e^{-az} > 0$. □

Lemma A8. *If* $b = 0$, $a = 0$, *then* $g > 0$ *and* $g' > 0$ *on* $(0,1)$.

Proof. Obvious from the expression of g, $g(z) = z$. □

Lemma A9. *If* $b \neq 0$, $a^2 = 4b$, *then* $g > 0$ *on* $(0,1)$. *Moreover, if* $a \leq 2$, *then* $g' > 0$ *on* $(0,1)$, *and, if* $a > 2$, $g' > 0$ *on* $(0, \frac{2}{a})$ *and* $g' < 0$ *on* $(\frac{2}{a}, 1)$.

Proof. Using that $g(z) = ze^{-\frac{a}{2}z}$, then $g'(z) = e^{-\frac{a}{2}z}\left(1 - \frac{a}{2}z\right)$, which is equal to zero at $z_0 = \frac{2}{a}$. If $a \leq 2$, then $g' > 0$ on $(0,1)$. If $a > 2$, then $g' > 0$ on $(0, \frac{2}{a})$ and $g' < 0$ in $(\frac{2}{a}, 1)$. However, in any case, $g > 0$ on $(0,1)$. □

Lemma A10. *Consider the case* $b \neq 0$, $a^2 > 4b$. *Then* $g > 0$ *on* $(0,1)$.

- *If one of the following properties holds:*

- $*$ $a = 0$,
- $*$ $a \neq 0$, *and* $b < 0$,
- $*$ $a < 0$, *and* $b > 0$,
- $*$ $a > 0$, $b > 0$, *and* $\frac{\beta}{\alpha} \geq e^{\alpha-\beta}$, *that is,*

$$a \leq \sqrt{a^2-4b}\,\frac{1+e^{\sqrt{a^2-4b}}}{e^{\sqrt{a^2-4b}}-1}, \tag{A15}$$

then $g' > 0$ *on* $(0,1)$.

- *On the other hand, if* $a > 0$, $b > 0$, *and* $\frac{\beta}{\alpha} < e^{\alpha-\beta}$, *that is,*

$$a > \sqrt{a^2-4b}\,\frac{1+e^{\sqrt{a^2-4b}}}{e^{\sqrt{a^2-4b}}-1},$$

then $g' > 0$ *on* $(0,\widehat{y})$ *and* $g' < 0$ *on* $(\widehat{y},1)$, *where* $\widehat{y} = \frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right) \in (0,1)$.

Proof. Now $g(z) = \dfrac{e^{\beta z}-e^{\alpha z}}{\beta-\alpha}$, and taking into account that $\alpha > \beta$, we obtain that $g > 0$ on $(0,1)$.

On the other hand, $g'(z) = \frac{1}{\beta-\alpha}\left(\beta e^{\beta z}-\alpha e^{\alpha z}\right)$, $z \in (0,1)$, where $\beta-\alpha < 0$.

If $a = 0$, then $b < 0$, $\alpha = \sqrt{-b}$ and $\beta = -\sqrt{-b}$, thus $\beta e^{\beta z}-\alpha e^{\alpha z} = -\sqrt{-b}e^{-\sqrt{-b}z}-\sqrt{-b}e^{\sqrt{-b}z} = -\sqrt{-b}(e^{\sqrt{-b}z}+e^{-\sqrt{-b}z}) < 0$, thus $g' > 0$ on $(0,1)$.

Suppose that $a \neq 0$. Note that, under the hypotheses of this lemma, α and β are necessarily different from zero. Indeed, if $\alpha = -\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b} = 0$, then $b = 0$, which is absurd, and analogously for β. We are interested in proving that $\beta e^{\beta z} < \alpha e^{\alpha z}$, for $z \in (0,1)$. We distinguish three cases, taking into account that $\alpha > \beta$:

Case 1: $\alpha > 0 > \beta$. If $a > 0$, then $\beta = -\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b}$ is trivially negative and $\alpha > 0$ means $\sqrt{(\frac{a}{2})^2-b} > \frac{a}{2}$, which is equivalent to $b < 0$.

If $a < 0$, $\alpha > 0$ is trivially valid and $\beta < 0$ if and only if $-\frac{a}{2} < \sqrt{\left(\frac{a}{2}\right)^2-b}$, which is the same as $b < 0$. Hence, if $a \neq 0$, $b < 0$, $\beta e^{\beta z} < \alpha e^{\alpha z}$, for every $z \in (0,1)$ and $g' > 0$ on $(0,1)$.

Case 2: $\alpha > \beta > 0$. In this case, $\beta e^{\beta z} < \alpha e^{\alpha z}$ if and only if $\frac{\beta}{\alpha} < e^{(\alpha-\beta)z}$, which holds for $\frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right) < z$. Since $0 < \frac{\beta}{\alpha} < 1$, then $\frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right) < 0$, hence $g' > 0$ on $(0,1)$.

Now, $-\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b} > -\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b} > 0$ implies that $a < 0$, $b > 0$ and conversely.

Case 3: $0 > \alpha > \beta$. In this case, $\beta e^{\beta z} < \alpha e^{\alpha z}$ if and only if $\frac{\beta}{\alpha} > e^{(\alpha-\beta)z}$, and this holds for $z < \frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right)$. If $\frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right) \geq 1$, that is, $\frac{\beta}{\alpha} \geq e^{\alpha-\beta}$, then $g' > 0$ on $(0,1)$.

Note that, in this case, $\frac{\beta}{\alpha} > 1$, thus $\widehat{y} = \frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right) > 0$. If $\frac{1}{\alpha-\beta}\ln\left(\frac{\beta}{\alpha}\right) < 1$ ($\frac{\beta}{\alpha} < e^{\alpha-\beta}$), then $g' > 0$ on $(0,\widehat{y})$ and $g' < 0$ on $(\widehat{y},1)$.

Moreover, $0 > -\frac{a}{2}+\sqrt{\left(\frac{a}{2}\right)^2-b} > -\frac{a}{2}-\sqrt{\left(\frac{a}{2}\right)^2-b}$ is equivalent to $a > 0$, $b > 0$.

Finally, $\widehat{y} \geq 1$, or $\frac{\beta}{\alpha} \geq e^{\alpha-\beta}$ can be written as:

$$\frac{-a-\sqrt{a^2-4b}}{-a+\sqrt{a^2-4b}} = 1-\frac{2\sqrt{a^2-4b}}{-a+\sqrt{a^2-4b}} \geq e^{\sqrt{a^2-4b}},$$

or

$$1-e^{\sqrt{a^2-4b}} \geq \frac{2\sqrt{a^2-4b}}{-a+\sqrt{a^2-4b}}.$$

Multiplying by $2\alpha = -a + \sqrt{a^2 - 4b} < 0$, we get:

$$\left(1 - e^{\sqrt{a^2-4b}}\right)\left(-a + \sqrt{a^2 - 4b}\right) \leq 2\sqrt{a^2 - 4b},$$

which is equivalent to (A15). $\square$

Lemma A11. *Consider $b \neq 0$, $a^2 < 4b$, and take $R := \sqrt{b - \frac{a^2}{4}} > 0$. Then $g > 0$ on $(0,1)$ if and only if $R \leq \pi$. Moreover:*

- *If one of the following conditions holds:*
  - $*$ *$R \leq \frac{\pi}{2}$ and $a \leq 0$,*
  - $*$ *$R < \frac{\pi}{2}$, and $0 < \frac{a}{2} \leq R\cot(R)$,*
  - $*$ *$\frac{\pi}{2} < R < \pi$, and $\frac{a}{2} \leq R\cot(R) < 0$,*

  then $g' > 0$ on $(0,1)$, and, thus, $g > 0$ on $(0,1)$.
- *If one of the following conditions holds:*
  - $*$ *$R = \frac{\pi}{2}$ and $a > 0$; or*
  - $*$ *$R < \frac{\pi}{2}$ and $\frac{a}{2} > R\cot(R)$; or*
  - $*$ *$R \in (\frac{\pi}{2}, \pi]$, and $a > 0$; or*
  - $*$ *$R \in (\pi, \frac{3\pi}{2})$, and $0 < \frac{a}{2} \leq R\cot(R)$,*

  then $g' > 0$ on $(0, z_0)$, and $g' < 0$ on $(z_0, 1)$, where $z_0 = \frac{1}{R}\arctan\left(\frac{2R}{a}\right)$.
- *If one of the following conditions hold:*
  - $*$ *$\frac{3\pi}{2} \geq R \geq \pi$, and $a < 0$; or*
  - $*$ *$\frac{\pi}{2} < R < \pi$, and $0 > \frac{a}{2} > R\cot(R)$,*

  then $g' > 0$ on $[0, z_1)$ and $g' < 0$ on $(z_1, 1)$, where

$$z_1 = \frac{1}{R}\left(\arctan\left(\frac{2R}{a}\right) + \pi\right).$$

- *If one of the following conditions hold:*
  - $*$ *$R \in (\pi, \frac{3\pi}{2})$, and $\frac{a}{2} > R\cot(R) > 0$; or*
  - $*$ *$R = \frac{3\pi}{2}$, and $a > 0$,*

  then $g' > 0$ on $(0, z_0)$, $g' < 0$ on (z_0, z_1), and $g' > 0$ on $(z_1, 1)$.
- *If $a \neq 0$, then the zeros of g' in $\mathbb{R}$ are*

$$z_k = \frac{1}{R}\left(\arctan\left(\frac{2R}{a}\right) + k\pi\right), \; k \in \mathbb{Z}.$$

  In consequence, if $R \leq \frac{\pi}{2}$, $a < 0$, then $g' > 0$ on $(0,1)$.
- *If $a = 0$, then $g' > 0$ in the interval $\left((2k-1)\dfrac{\pi}{2R}, (2k+1)\dfrac{\pi}{2R}\right)$, for $k \in \mathbb{Z}$ even, and $g' < 0$ in that interval for $k \in \mathbb{Z}$ odd. In consequence, if $a = 0$ and $R \leq \frac{\pi}{2}$, then $g' > 0$ on $(0,1)$.*

Proof. The expression of g is given by $g(z) = e^{-\frac{a}{2}z}\dfrac{\sin Rz}{R}$, $\forall z$. If $R \leq \pi$, then $0 < Rz < \pi$, for every $z \in (0,1)$, and $g > 0$ on $(0,1)$. In fact, it is clear that $g > 0$ on $(0,1)$ if and only if $R \leq \pi$.

The derivative of g is given by:

$$g'(z) = e^{-\frac{a}{2}z}\left\{\cos Rz - \frac{a}{2R}\sin Rz\right\}, \ \forall z.$$

Note that the study of the sign of g' is similar to the study of the sign of h_2' in Lemma A6, taking $\widetilde{N} = \frac{a}{2R}$ instead of N (conditions are obtained just by considering $c = 0$ in the conditions in Lemma A6). □

We remark that the sign of g' coincides with the sign of ϕ, given by

$$\phi(z) = \cos Rz - \frac{a}{2R}\sin Rz.$$

Remark A4. *For the case $a = 0$ ($b > 0$), we obtain that $g'(z) = \cos Rz > 0$ in $\left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right)$ for k even, and $g' < 0$ in $\left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right)$ for k odd.*

Therefore, if $g((2k+1)\frac{\pi}{2R}) \geq 0$, for every k odd with $(2k+1)\frac{\pi}{2R} \in (0,1)$, then $g \geq 0$ on $(0,1)$. That is, we have to check the sign of function g at the points $(2k+1)\frac{\pi}{2R}$, for k odd with $\frac{-1}{2} < k < \frac{R}{\pi} - \frac{1}{2}$, that is, $k = 1, 3, \ldots$ with $k < \frac{R}{\pi} - \frac{1}{2}$.

Remark A5. *Consider $a^2 < 4b$. If $a \neq 0$ ($b > 0$), the zeros of g' coincide with the zeros of ϕ, that is, the points*

$$z_k = \frac{1}{R}\left(\arctan\left(\frac{2R}{a}\right) + k\pi\right) \in \left((2k-1)\frac{\pi}{2R}, (2k+1)\frac{\pi}{2R}\right), \ k \in \mathbb{Z}.$$

Therefore, the situation is analogous to the study made in Remark A2, just by replacing N by $\widetilde{N} = \frac{a}{2R}$. Function ϕ is also $\frac{2\pi}{R}$-periodic.

In consequence,

* *If $a > 0$, then $\phi > 0$ on (z_{2k-1}, z_{2k}), for $k \in \mathbb{Z}$, and $\phi < 0$ on (z_{2k}, z_{2k+1}), for $k \in \mathbb{Z}$.*
* *If $a < 0$, then $\phi > 0$ on (z_{2k}, z_{2k+1}), for $k \in \mathbb{Z}$, and $\phi < 0$ on (z_{2k-1}, z_{2k}), for $k \in \mathbb{Z}$.*

Summarizing, if one of the following conditions holds:

- *$a > 0$, and $g(z_k) \geq 0$, for every k odd with $z_k \in (0,1)$; or*
- *$a < 0$, and $g(z_k) \geq 0$, for every k even with $z_k \in (0,1)$,*

then $g \geq 0$ on $(0,1)$. If the inequalities above are strict, then g is strictly positive on $(0,1)$.

The points z_k of our interest are the following:

- *If $a > 0$:*
  * *k odd with $1 \leq k \leq \frac{R}{\pi} - \frac{1}{2}$; and*
  * *if $\frac{a}{2} > R\cot(R) > 0$, $k \geq 1$ odd with $\frac{R}{\pi} - \frac{1}{2} < k < \frac{R}{\pi}$, that is, $R \in (k\pi, (2k+1)\frac{\pi}{2})$ with $k \geq 1$ odd.*
- *If $a < 0$:*
  * *k even with $1 < k \leq \frac{R}{\pi}$; and*
  * *if $R\cot(R) < \frac{a}{2} < 0$, $k > 1$ even with $\frac{R}{\pi} < k < \frac{R}{\pi} + \frac{1}{2}$, that is, $R \in ((2k-1)\frac{\pi}{2}, k\pi)$, with $k > 1$ even.*

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